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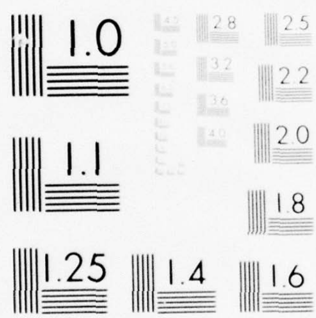

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REPORT R-762 MARCH, 1977 UILU-ENG 77-2209

COORDINATED SCIENCE LABORATORY

SENSITIVITY OF LOWER ORDER OBSERVERS

ERWIN DE SA

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The problem of sensitivity of dynamic systems to changes in parameters ~~has~~ is of ~~acquired~~ great importance in modern technology. Sensitivity is important because in physical realization of ~~systems of~~ automatic control systems, changes in the parameters always are encountered as a result of aging of the elements, the effects of the external environment, interactions with other systems, etc. Furthermore, contemporary automatic control systems often are realized as systems of variable structure where the possibility of varying the parameters of the system is specially introduced to obtain adaptation (cont on 1473B)

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20. ABSTRACT (continued)

properties, as, for example, in systems of optimal control or in self-learning systems.

In all these cases, the question arises as to how the behavior of the system changes when changes occur in certain of its parameters. One such system of interest is the lower order observer or lower order estimator.

The determination of changes in the performance of the lower order observer to changes in parameters is of great importance in engineering analysis and design. Given the fact that the output of the lower order observers are physically not accessible, a study of the changes of these unaccessible states to changes in system parameters are helpful in determining the properties of a system. Besides this, observers are used for adaptive schemes in which the sensitivity functions are needed.

The significance of using lower order observers instead of a full order observer, when a few states are needed to be estimated, is of an economical nature. Sometimes it is needed that the estimated state be highly non-sensitive to parameter variations, the economic reasons are then superseded by sensitivity. Therefore, a comparison in sensitivity is needed between the lower order, and full order, observers.

SENSITIVITY OF LOWER ORDER OBSERVERS

BY

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B.E., University of Karachi, Pakistan, 1974

THESIS

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for the degree of Master of Science in Electrical Engineering
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TABLE OF CONTENTS

1. INTRODUCTION

The problem of sensitivity of dynamic systems to changes in parameters has acquired great importance in modern technology. Sensitivity is important because in physical realization of systems of automatic control, changes in the parameters always are encountered as a result of aging of the elements, the effects of the external environment, interactions with other systems etc. Furthermore, contemporary automatic control systems often are realized as systems of variable structure where the possibility of varying the parameters of the system is specially introduced to obtain adaptation properties as, for example, in systems of optimal control or in self-learning systems.

In all these cases the question arises as to how the behavior of the system changes when changes occur in certain of its parameters. One such system of interest is the lower order observer or lower order estimator.

The determination of changes in the performance of the lower order observer to changes in parameters is of great importance in engineering analysis and design. Given the fact that the output of the lower order observers are physically not accessible, a study of the changes of these unaccessible states to changes in system parameters are helpful in determining the properties of a system. Besides this, observers are used for adaptive schemes in which the sensitivity functions are needed.

The significance of using lower order observers instead of a full order observer, when a few states are needed to be estimated, is of an economical nature. Sometimes it is needed that the estimated state be

highly nonsensitive to parameter variations, the economic reasons are then superseded by sensitivity. Therefore a comparison in sensitivity is needed between the lower order, and full order, observers.

In order to study this behavior the sensitivity functions or the parameter influence coefficients, of the estimated states of the system with respect to system parameters, will have to be generated. If the system is directly differentiated with respect to the parameters as in [9], resulting in differential equations of very high dimensions. To avoid this method, the method of generating sensitivity functions as proposed by Wilkie and Perkins in [11,12] is adapted to reduce the model of the system.

Chapter two deals entirely with the development of the sensitivity model. As an example, the Luenberger Observer has been used. The low order sensitivity model is determined and certain schemes have been set up to help design this model. Chapter three deals with the experimentation of some practical examples. The examples as, for example a d-c motor and a thermometer are used in order to give the reader a physical feeling for the analysis of these systems. Chapter four deals with the conclusions drawn from the experimental results of Chapter three.

Finally, this analysis will shed some light on the properties of the lower order observers. In no way are the conclusions in this thesis applicable to each and every lower order observer, but it is assumed that the conclusions drawn in this thesis can be attributed to lower order observers in general.

2. SENSITIVITY MODEL OF THE LOWER ORDER OBSERVER

2.1. Introduction

In this chapter, the sensitivity model for the lower order observer will be developed. It is of considerable practical and theoretical interest to obtain the trajectory sensitivity functions. In the case of this study, the functions will be used to study the advantages or disadvantages of the lower order observer, which is carried out in Chapter 3. From a practical point of view, low-order realizations are required for efficient use of equipment in simulations and for reasons of accuracy.

In this chapter we shall show that all the sensitivity functions can be generated from a n-dimensional sensitivity model, or under certain conditions, from a (n-r)-dimensional model.

An overview of sensitivity is available in [3,4]. The sensitivity model developed in this chapter is a direct application of the model proposed by Wilkie and Perkins in [11,12].

2.2. Lower Order Observers

The state variables of a system are sometimes needed for various purposes, the common one being state feedback. When designing a feedback, it is desired that all the states of the system are available. This desired situation often does not hold in practice, however, either because the state variables are not accessible for direct measurement or because the number of measuring devices is limited. Therefore, in order to stabilize, decouple and to optimize a system that has a limited number of states available, a reasonable substitute for the state vector has to be found. The dynamical system that constructs an approximation of the state vector is called a state estimator or an observer.

There are many different kinds of observers. A textbook summary of this subject is given in [2]. Generally there are two types of observers. The n-dimensional or full order observer and the lower order observer. The n-dimensional full order observer estimates all the states of the system. Sometimes this is unnecessary, as some of the states may be obtained exactly from measured outputs. In such cases a lower order observer is used to estimate only those states that are required.

The lower order observer can be designed in many ways. A textbook treatment of observers may be found in [2]. In this study we shall consider the Luenberger Observer[8,13]. We shall use the circumflex over a variable to denote an estimate of the variable. For example, $\hat{x}_1$ is an estimate of x_1; $\hat{x}_2$ is an estimate of x_2.

Consider the time-invariant, n-dimensional system described by the state equations

$$\dot{x} = Ax + Bu \tag{2.2.1}$$

$$y = Cx \tag{2.2.2}$$

The output y is r-dimensional.

The matrices A, B and C are assumed to be known. The problem is to generate all state variables from the available input u and output y.

The equation of the Luenberger Observer as given in [5,7] are

$$\dot{Z} = PZ + \theta y + Ru \tag{2.2.3}$$

$$\hat{x}_a = Z + Ky \tag{2.2.4}$$

where

$\hat{x}_a$ is the $(n-r) \times 1$ state vector

Z is the $(n-r) \times 1$ state vector

P is an $(n-r) \times (n-r)$ real constant matrix

θ is an $(n-r) \times r$ real constant matrix

R is an $(n-r) \times n$ real constant matrix

K is an $(n-r) \times r$ real constant matrix

Assume C has rank r. There is no loss in generality in this assumption. If rank $C < r$, some outputs are linearly dependent on others and can be eliminated. Then there exists an $(n-r)xn$ matrix N_1 such that the transformation N in

$$\begin{bmatrix} x_a \\ x_b \end{bmatrix} \triangleq Nx \triangleq \begin{bmatrix} N_1 \\ C \end{bmatrix} x \tag{2.2.5}$$

is non-singular.

Then

$$NAN^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad NB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{2.2.6}$$

Therefore

$$\dot{x}_a = A_{11}x_a + A_{12}x_b + B_1u \tag{2.2.7}$$

$$\dot{x}_b = A_{21}x_a + A_{22}x_b + B_2u \tag{2.2.8}$$

The appropriate choices of matrices for (2.2.3) and (2.2.4) can then be shown to be

$$P = A_{11} - KA_{21} \tag{2.2.9}$$

$$\theta = (A_{11} - KA_{21})K + (A_{12} - KA_{22}) \tag{2.2.10}$$

$$R = B_1 - KB_2 \tag{2.2.11}$$

The matrix K is selected so that the eigenvalues of $A_{11} - KA_{21}$ are the desired eigenvalues, and make the error, decay to zero. Figure 2.1 is the block diagram of the Luenberger Observer.

2.3. Sensitivity

The system given by (2.2.1) can be dependent on many parameters. In general let the parameter vector effecting system (2.2.1) be given by g, where vector g is N-dimensional.

A variation in the parameter g will produce a variation in the solution, which, to first order, is given by

$$\delta \hat{x}_a(t,g) \approx \sum_i^N \frac{\delta \hat{x}}{\delta g_j} dg_j. \qquad j = 1,2,\ldots,N \tag{2.3.1}$$

The partial derivatives

$$\frac{\delta \hat{x}_{a\ell}}{\delta g_j} \triangleq \zeta_{\ell,j} \qquad \ell = 1,2,\ldots,n-r \qquad j = 1,2,\ldots,N \tag{2.3.2}$$

are called the estimated state trajectory sensitivity functions.

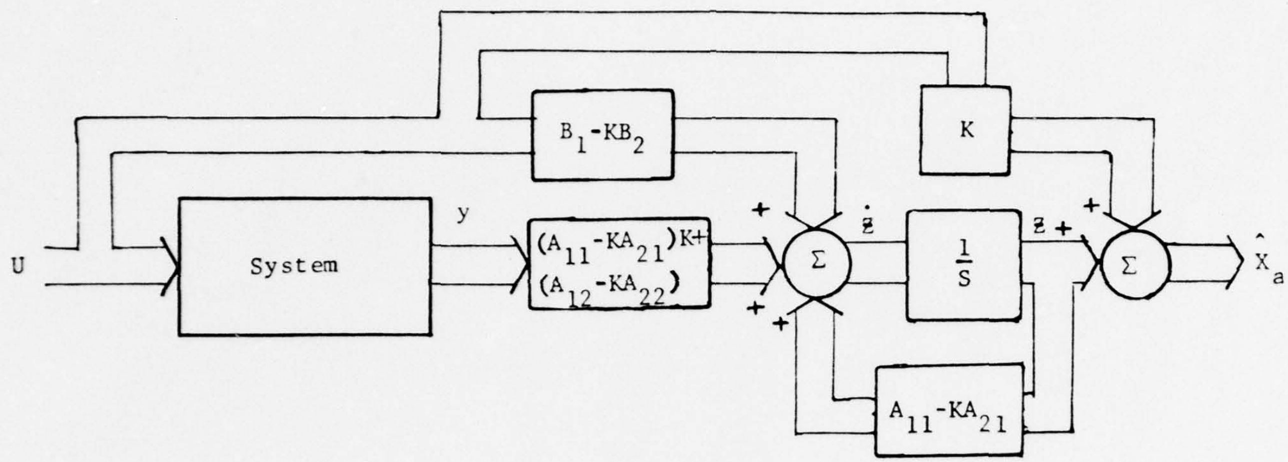

Figure 2.1 Block diagram of the Luenberger Observer.

These sensitivity functions are of prime importance in determining the effect of parameter variations on the effectiveness of the observer in estimating the state.

The linear time-invariant, n-dimensional system described by (2.2.1) is now written as

$$\dot{x}(g) = A(g)x(g) + B(g)u \tag{2.3.3}$$

$$y(g) = C\,x(g) \tag{2.3.4}$$

where g is a N-dimensional parameter vector.

Considering the lower order observer given by (2.2.3) and (2.2.4), it must be noted that the matrices P, θ, R and K are constants and are desired at a nominal value of the parameter g, given as g^0.

Thus the functional dependence of (2.2.3) and (2.2.4)on g is

$$\dot{Z}(g) = PZ(g) + \theta y(g) + Ru \tag{2.3.5}$$

$$\hat{x}(g) = Z(g) + Ky(g) \tag{2.3.6}$$

2.4. Sensitivity Model for the Lower Order Observer in Open-loop

In the introduction to this chapter, the significance of reduced models was explained. The sensitivity functions for linear time-invariant systems may be obtained, in principle, by solving the linear equations with literal parameters and differentiating the result. This procedure is extremely tedious and unilluminating. A more useful approach, as shown by Wilkie and Perkins in [11,12], is to obtain differential equations for the trajectory sensitivity functions. Furthermore, the structure of the

resulting simulation itself often provides helpful insight into the system behavior with respect to the parameters. The Wilkie-Perkins low order sensitivity model will now be developed for the observer given by (2.2.3) and (2.2.4). It must be noted that the sensitivity functions needed are the estimated state sensitivity functions only and not the sensitivity functions of the available states. We shall now consider the single input case of system (2.2.1).

Now consider (2.2.7). It is a (n-r)-dimensional equation and is the differential equation of all of the states that have to be estimated. Then, the sensitivity model for only the lower order observer will be a (n-r)-dimensional model. It is seen that the input to this model are the sensitivity functions of the output of system (2.2.1), therefore, we will be required to generate the output sensitivity functions of system (2.2.1).

In order to accomplish this, we substitute $\hat{x}_a$ for x_a in (2.2.5) and get

$$\tilde{x} = N^{-1} \begin{bmatrix} \hat{x}_a \\ \\ x_b \end{bmatrix} \tag{2.4.1}$$

The vector is similar to the state vector x and therefore we can write

$$\dot{\tilde{x}} = A\tilde{x} + bu \tag{2.4.2}$$

$$y = C\tilde{x} \tag{2.4.3}$$

where b is a n $\times$ 1 matrix. Note that system (2.4.2), (2.4.3) has matrix A which is similar to system (2.2.1) and therefore (2.4.2), (2.4.3) is considered to be an open-loop system. We shall now proceed to develope an open-loop sensitivity model for (2.4.2), (2.4.3).

In [11] it has been shown that for the controllable system (2.4.2) and (2.4.3) a non-singular transformation

$$\tilde{x} = Tq \tag{2.4.4}$$

always exists, such that

$$\dot{q} = \tilde{A}q + \tilde{b}u \tag{2.4.5}$$

where

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_n \end{bmatrix}, \quad \tilde{b} = T^{-1}b \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{2.4.6}$$

The matrix T can easily be found as shown in [3]. The elements $\alpha_1, \alpha_2, \dots, \alpha_n$ in matrix $\tilde{A}$ are the coefficients in the characteristic equation.

It is evident that A, b and T are a function of the parameter vector g. The sensitivity functions for these parameters will now be found.

It can easily be shown that (2.4.5) can be written as

$$\dot{q}(t,\alpha) = \tilde{A}(\alpha)q(t,\alpha) + \tilde{b}u(t) \tag{2.4.7}$$

Taking the partial deravative of both sides of (2.4.7) with respect to α and interchanging the order of the differential equation we get

$$\frac{d}{dt}\frac{\delta q}{\delta \alpha_i}(t,\alpha) = \frac{\delta \tilde{A}(\alpha)}{\delta \alpha_i} q(t,\alpha) + \tilde{A}\,\frac{\delta q(t,\alpha)}{\delta \alpha_i} \qquad (2.4.8)$$

$$i = 1,2,\ldots,n$$

Let
$$\frac{\delta q(t,\alpha)}{\delta \alpha_i} = h_i, \quad i=1,2,\ldots,n \qquad (2.4.9)$$

such that the initial conditions are

$$h_0^i \triangleq h_i(t,\alpha)\Big|_{t=0} = \frac{\delta q(t,\alpha)}{\delta \alpha_i}\Big|_{t=0} \qquad (2.4.10)$$

Assuming appropriate continuity conditions we interchange the order of the partial with respect to $\tilde{A}$ and substitute t=0.

Then,

$$h_0^i = \frac{\delta}{\delta \alpha_i}\left[q(t,\alpha)\Big|_{t=0}\right] = \frac{\delta q_0}{\delta \alpha_i} \qquad (2.4.11)$$

From (2.4. 8) and (2.4. 9) we get

$$\dot{h}_i(t,\alpha) = \tilde{A}(\alpha)h_i(t,\alpha) + \frac{\delta \tilde{A}(\alpha)}{\delta \alpha} q(t,\alpha) \qquad (2.4.12)$$

From the differential equations given by (2.4.12) the sensitivity functions for the estimated states can be easily found. The subject is studied thoroughly in Section 2.5.

There are two important properties of the sensitivity model. As the model is similar to the one described by Wilkie and Perkins in [11,12] the proof of these properties will be omitted, but the properties will be stated for convenience.

Total Symmetry Property

The sensitivity functions defined by (2.4.9) is given as

$$h_i = \frac{\delta q(t,\alpha)}{\delta \alpha_i} \qquad i = 1,2,\ldots,n$$

and a sensitivity matrix $[h_{ij}]$ as

$$[h_{i,j}] \triangleq [h_1 \; h_2 \; h_2 \; . \; . \; . \; h_n] \qquad (2.4.13)$$

The sensitivity matrix $[h_{i,j}]$ has the following total symmetry property

$$h_{i,j}(t) = h_{i-1,j+1}(t) \; , \qquad \begin{array}{l} i = 1,2,\ldots,n \\ j = 1,2,\ldots,n-1 \end{array} \qquad (2.4.14)$$

Then all the elements along the antidiagonals of $[h_{i,j}]$ are equal as shown in (2.4.15).

$$h_{i,j} = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,n} \\ h_{1,2} & \cdot & \cdot & & h_{2,n} \\ h_{1,3} & & & & h_{n-1,n} \\ h_{i,n} & h_{2,n} & \cdots & h_{n-1,n} & h_{n,n} \end{bmatrix} \qquad (2.4.15)$$

Complete Simultaneity Property

All the sensitivity functions $\frac{\delta q_i}{\delta \alpha_i}$, i, j = 1,2,...,n, can be obtained as linear combinations of the states of the sensitivity model (2.4.12) and the system states q.

Thus far we have developed a n-dimensional sensitivity model. The important point of this model is the fact that all the sensitivity functions can be generated by this model. If for example we have j parameters in the system, then all the (n-r) $\times$ j sensitivity functions can be generated by the n-dimensional model. It is important to state at this point that, by carefully choosing $\frac{\delta A}{\delta \alpha}$, the input to the sensitivity model can be an available state of system (2.2.1).

2.5. Sensitivity Functions

In Section 2.4 a sensitivity model was developed for the observer, which is given by (2.4.12). We will now show how the estimated state sensitivity functions will be generated from one model of n-dimension.

From the transformations (2.2.5) and (2.4.4) we have

$$\hat{x}_a = N_1 Tq \tag{2.5.1}$$

Note that T is a function of the parameter vector g.

Taking the partial of both sides with respect to α we get

$$\frac{\delta \hat{x}_{a\ell}(t,\alpha)}{\delta \alpha_i} = [N_1 T] h_i(t,\alpha) + \frac{\delta [N_1 T(g)]}{\delta \alpha_i} q(t,\alpha) \qquad \begin{array}{l} \ell = 1,2,\ldots,n-r \\ i = 1,2,\ldots,n \end{array} \tag{2.5.2}$$

From (2.3.2) we have

$$\frac{\delta \hat{x}_{a\ell}}{\delta gj} \triangleq \zeta_{\ell,j} \qquad \begin{array}{l} \ell = 1,2,\ldots,n-r \\ j = 1,2,\ldots,N \end{array} \tag{2.5.3}$$

then

$$\frac{\delta \hat{x}_{a\ell}}{\delta \alpha_i} \cdot \frac{\delta \alpha_i}{\delta g_j} \triangleq \zeta_{\ell,j} \qquad \begin{array}{l} \ell = 1,2,\ldots,n\text{-}r \\ j = 1,2,\ldots,N \end{array} \tag{2.5.4}$$

Substituting (2.5.2) in (2.5.4) we have

$$\left\{[N_1T]h_{\ell,i}(t,\alpha) + \frac{\delta[N_1T]}{\delta \alpha_i}\, q(t,\alpha)\right\} \cdot \frac{\delta \alpha_i}{\delta g_j} \triangleq \zeta_{\ell,j} \qquad \begin{array}{l} i = 1,2,\ldots,n \\ \ell = 1,2,\ldots,n\text{-}r \\ j = 1,2,\ldots,N \end{array} \tag{2.5.5}$$

As C is independent of g, then N_1 is also independent of g. We can interchange h $\frac{\delta\alpha}{\delta g}$ with $\frac{\delta\alpha}{\delta g}$ h. Therefore

$$[N_1T]\left[\frac{\delta \alpha_i}{\delta g_j}\right]h_{\ell,i}(t,\alpha) + [N_1]\left[\frac{\delta T(g)}{\delta g_j}\right]q(t,\alpha) \triangleq \zeta_{\ell,j} \qquad \begin{array}{l} i = 1,2,\ldots,n \\ \ell = 1,2,\ldots,n\text{-}r \\ j = 1,2,\ldots,N \end{array} \tag{2.5.6}$$

The equation (2.5.6) gives all the sensitivity functions, (n-r) × N. Thus we see that a n-dimensional model gives all the (n-r) × N estimated state sensitivity functions. It is very easy to show how these functions are generated. The states $h_i(t,\alpha)$ are derived from the sensitivity model (2.4.12). $q(t,\alpha)$ is generated by transformation (2.4.4). The $\left[\frac{\delta \alpha_i}{\delta g_j}\right]$ are easily calculated as the functions α_i and g_j are known and can be easily calculated on a computer. For example if $g_1 = \alpha_1$ then (2.5.6) will be given as

$$[N_1T]h_{\ell,1} + [N_1]\left[\frac{\delta T(g)}{\delta \alpha_1}\right]q(t,\alpha) \triangleq \zeta_{\ell,1} \qquad \ell = 1,2,\ldots,n\text{-}r \tag{2.5.7}$$

In the same way all the sensitivity functions can be found. It is very important to note here that in the case of single input, the maximum number of parameters are n^2+n.

It has been shown that the matrix T can be calculated by the relations [1,14]

$$T = [t_1 \ t_2 \ . \ . \ . \ t_n] \tag{2.5.8}$$

where

$$\begin{aligned} t_n &= b \\ t_{n-1} &= At_n + \alpha_n t_n \\ t_{n-2} &= At_{n-1} + \alpha_{n-1} t_n \\ &\cdot \\ &\cdot \\ &\cdot \\ t_1 &= At_2 + \alpha_2 t_n \end{aligned} \tag{2.5.9}$$

Differentiating (2.5.9) with respect to g_j we get

$$\frac{\delta T}{\delta g_j} = \left[\frac{\delta t_1}{\delta g_j} \ . \ . \ . \ . \ . \ \frac{\delta t_n}{\delta g_j}\right] \tag{2.5.10}$$

$$\begin{aligned} \frac{\delta t_n}{\delta g_j} &= \frac{\delta b}{\delta g_j} \\ \frac{\delta t_{n-1}}{\delta g_j} &= \frac{\delta A}{\delta g_j} t_n + \frac{\delta \alpha_n}{\delta g_j} b + A \frac{\delta t_n}{\delta g_j} + \alpha_n \frac{\delta b}{\delta g_j} \\ \frac{\delta t_{n-2}}{\delta g_j} &= \frac{\delta A}{\delta g_j} t_{n-1} + A \frac{\delta t_{n-1}}{\delta g_j} + \frac{\delta \alpha_{n-1}}{\delta g_j} b + \alpha_{n-1} \frac{\delta b}{\delta g_j} \\ &. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ \frac{\delta t_1}{\delta g_j} &= \frac{\delta A}{\delta g_j} t_2 + A \frac{\delta t_2}{\delta g_j} + \frac{\delta \alpha_2}{\delta g_j} b + \alpha_2 \frac{\delta b}{\delta g_j} \end{aligned} \tag{2.5.11}$$

Thus, if the functional dependence of the system matrix A is assumed known (so that $\frac{\delta A}{\delta g_j}$ is known) and also $\left[\frac{\delta \alpha_n}{\delta g_j}\right]$ is known. The $\left[\frac{\delta T(g)}{\delta g_j}\right]$ can be easily calculated. Experience has shown that when using computers the time needed for the algebraic calculations associated with the transformation to canonic form is much less than the time required for the additional integrations needed in direct implementation of the model. Figure 2.2 is the block diagram of the sensitivity model, as proposed in Section 2.5. A few examples have been worked out in Chapter 3.

2.6. Sensitivity Model for the Lower Order Observer in a Feedback Loop

It has been mentioned before that the lower order observer is used frequently for state feedback. It would be appropriate, then to develop a model for the closed-loop sensitivity functions. Using the same technique as developed in Section 2.5, we will now show how to develop this model.

Consider (2.4.7). Let the feedback be given by

$$u = - u_a + u_b \tag{2.6.1}$$

where

$$u_a = \phi q \tag{2.6.2}$$

Let matrix $\phi = [\phi_1 \; \phi_2 \; \phi_3 \; . \; . \; . \; \phi_n] = FT$, where F is a constant matrix and T is the transformation matrix from (2.4.4), dependent on the parameter vectors α from A and β from matrix b. The input u_a is then, a function of the parameter vectors α and β. Let the input u_b be a function of time. Note that the coefficients of the parameter vector β are obtained from the matrix b. A textbook treatment of state feedback pole placement may be found in [10].

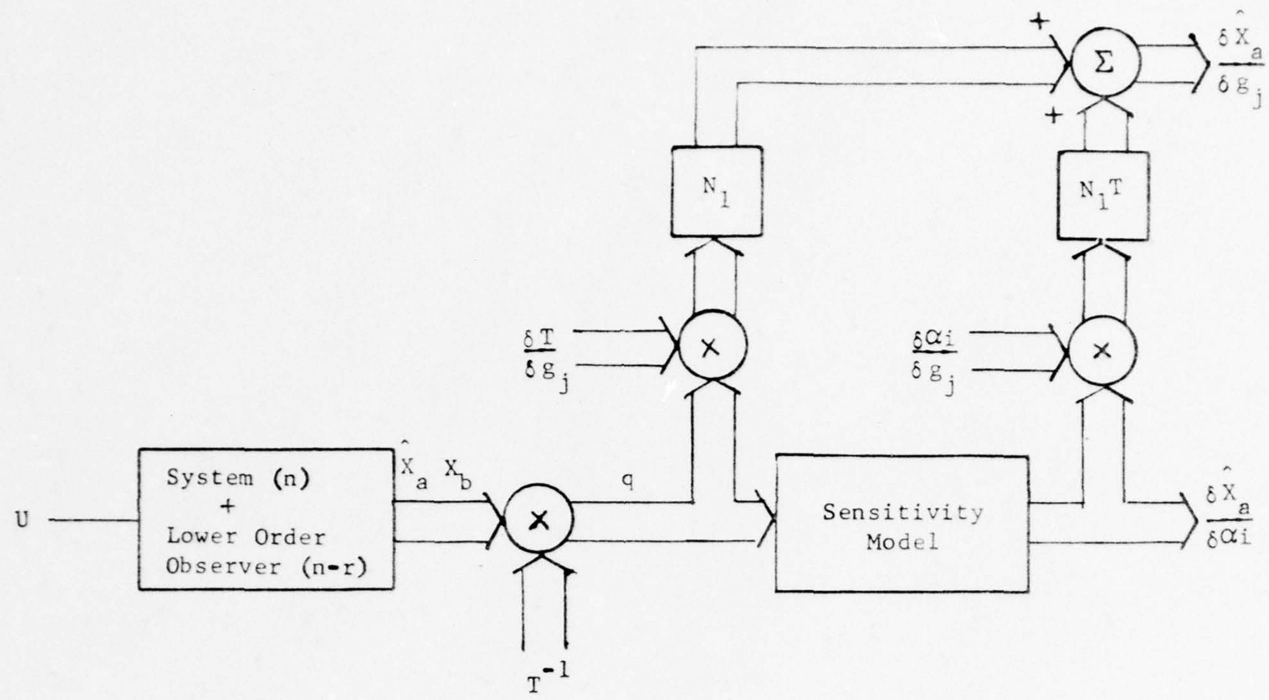

Figure 2.2 Block diagram for generating all the open-loop sensitivity functions.

Combining (2.4.7) and (2.6.1) we get

$$\dot{q}(t,\alpha,\beta) = \left(\tilde{A}(\alpha) - b\phi(\alpha,\beta)\right)q(t,\alpha,\beta) + bu_b(t) \tag{2.6.3}$$

Equation (2.6.3) can then be written as

$$\dot{q}(t,\omega) = W(\omega)q(t,\omega) + bu_b(t) \tag{2.6.4}$$

where

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -\omega_1 & -\omega_2 & -\omega_3 & \cdots & -\omega_n \end{bmatrix} \tag{2.6.5}$$

and

$$\begin{aligned} \omega_1 &= \alpha_1 + \phi_1 \\ \omega_2 &= \alpha_2 + \phi_2 \\ \omega_3 &= \alpha_3 + \phi_3 \\ &\cdot \\ &\cdot \\ &\cdot \\ \omega_n &= \alpha_n + \phi_n \end{aligned} \tag{2.6.6}$$

The parameters $\omega_1, \omega_2, \ldots, \omega_n$ are the coefficients of the parameter vector ω.

Let $$\frac{\delta q(t,\omega)}{\delta \omega_i} \triangleq S_i \qquad i = 1,2,\ldots,n \tag{2.6.7}$$

such that the initial conditions are

$$S_0^i \overset{\Delta}{=} S_i(t,\omega)\Big|_{t=0} = \frac{\delta q(t,\omega)}{\delta \omega_i}\Big|_{t=0} \tag{2.6.8}$$

also

$$S_0^i = \frac{\delta}{\delta \omega_i}\left[q(t,\omega)\Big|_{t=0}\right] = \frac{\delta q_0}{\delta \omega_i}\,. \tag{2.6.9}$$

Using the same techniques as used in Section 2.5 we can easily show that

$$\dot{S}_i(t,\omega) = W(\omega)S_i(t,\omega) + \frac{\delta W(\omega)}{\delta \omega_i}\,q(t,\omega). \tag{2.6.10}$$

Equations (2.6.10) and (2.4.12) are alike excepting that, (2.6.10) is dependent on a parameter vector ω and (2.4.12) on a parameter vector α. Similarly, it can be shown that (2.6.10) also has the two properties of (2.4.12) and therefore we can generate all the sensitivity functions from this n-dimensional model.

The equations of the sensitivity functions are then given as

$$\pi_{\ell,j} \overset{\Delta}{=} [N_1 T]\left[\frac{\delta \omega_i}{\delta g_j}\right] S_{\ell,i}(t,\omega) + [N_1]\left[\frac{\delta T(g)}{\delta g_j}\right] q(t,\omega) \tag{2.6.11}$$

$$i = 1,2,\ldots,n$$
$$\ell = 1,2,\ldots,(n-r)$$
$$j = 1,2,\ldots,r$$

These are the equations of the sensitivity functions. It must be noted that the closed-loop model and the open-loop model are similar, excepting that, whereas the open-loop model has parameter vector α, the closed-loop model has parameter vector ω. Using the same discussion as before we can easily compute $\left[\frac{\delta \omega_i}{\delta g_j}\right]$, $\left[\frac{\delta T(g)}{\delta g_j}\right]$ is computed as shown before.

This concludes our discussion of the sensitivity model. In the next chapter we shall generate the sensitivity functions of a few examples and analyze them.

3. ANALYSIS OF THE SENSITIVITY OF LOWER ORDER OBSERVERS

3.1. Introduction

When designing a lower order observer it is very important to know how good the estimation will be. One of the fundamental requirements is to reduce the error between the output of the system and the estimated states, finally decaying to zero in finite time. In Section 2.2 we have seen how this is achieved by selecting the matrix K, in other words by selecting the eigenvalues of the lower order observer. This approach to the problem is fine, assuming that nothing else effects the estimated states. In practice, however this is not true. One of the reasons is that the parameters of a system are prone to variations, thus affecting the estimated states. The error then, is no longer only dependent on the eigenvalues of the lower order observer but also on the parameters of the system. If the error for a particular estimated state is required to remain within fixed limits and to decay to zero in finite time then there may be a set of eigenvalues of the lower order observer that will accomplish this. But there may be one single parameter in the system that could drive the error beyond the limits, thus eliminating the possibility of using that particular lower order observer.

It has been mentioned before that lower order observers are used extensively for the purpose of state feedback. Many investigators have mentioned, that the sensitivity of a system improves when using feedback. It is therefore intriguing to study the sensitivity of lower order observers in a feedback loop. Another aspect that is worthy to study is that, given a set of allowable eigenvalues of the lower order observer, which eigenvalues

will cause the lower order observer to be least dependent on the parameters, that is, effects of the eigenvalues of a lower order observer on its sensitivity. This is a very important element in optimizing the best eigenvalues in order to design the best observer.

It was mentioned earlier that one of the reasons that lower order observers were preferred to full order observers was economics. Sometimes this could prove untrue, when sensitivity supersedes economics and a highly insensitive observer has to be designed. The question then arises, of how do lower order observers and full order observers compare in sensitivity. This is another point that needs investigation.

Thus, we see that a study of the sensitivity functions of the estimated states will give us an insight into some of the aspects of designing a better lower order observer or replacing it by a full order observer. In this chapter we shall investigate the above mentioned points. Some practical examples will be used to illustrate the subject matter. It must be noted here that the analysis will cover only the topics mentioned, even though there may be many more related aspects that need to be investigated.

Examples 1, 2, 3, and 4 generate the open-loop sensitivity functions and thus will be used for open-loop sensitivity analysis. Examples 5, 6, and 7 are used for the closed-loop analysis of the lower order observer sensitivity functions.

3.2. Example No. 1

The equations of an armature coupled d-c motor is given by the equations

$$\left(\frac{L_m}{K_T} JS^3 + \frac{L_m B + R_m J}{K_T} S^2 + \frac{R_m B + K_b K_T}{K_T} S\right)\theta_m = e_a \tag{3.2.1}$$

where θ_m is the angle of displacement of the motor

e_a input voltage to the armature

L_m inductance of the armature

R_m resistance of the armature

J moment of inertia of motor plus load

B damping coefficient

K_T and K_b proportionality constants.

The nominal value of these parameters are given by

$$\begin{aligned}
L_m^o &= 1 \text{ henry} \\
R_m^o &= 2 \text{ ohms} \\
J^o &= 1.125 \text{ Kg-m}^2 \\
B^o &= 1.125 \text{ newton/(meter/second)} \\
K_T^o &= 2.25 \text{ newton-m/amp or volt-sec/radian} \\
K_b^o &= 1 \text{ volts-meter-sec/weber-rad.}
\end{aligned} \tag{3.2.2}$$

Equation (3.2.1) can be put into the form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\alpha_2 & -\alpha_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \beta_3 \end{bmatrix} u, \quad y = x_1 \tag{3.2.3}$$

where

$$\alpha_2^o = \frac{R_m^o B_m^o + K_b^o K_T^o}{L_m J} = \frac{2 \times 1.125 \times 1 \times 2.25}{1.125 \times 1} \approx 4 \tag{3.2.4}$$

$$\alpha_3^o = \frac{L_m^o B^o + R_m^o J^o}{L_m J} = \frac{1.125 \times 1 + 1.125 \times 2}{1.125 \times 1} = 3 \tag{3.2.5}$$

$$\beta_3^o = \frac{K_T^o}{L_m^o J^o} = \frac{2.25}{1.125 \times 1} = 2 \tag{3.2.6}$$

The eigenvalues of the system are $0, -\frac{3}{2} + j\frac{\sqrt{7}}{2}, -\frac{3}{2} - j\frac{\sqrt{7}}{2}$.

There are six parameters in this system. They are

$$\begin{aligned} q_1 &= L_m \\ q_2 &= R_m \\ q_3 &= J \\ q_4 &= B \\ q_5 &= K_T \\ q_6 &= K_b \end{aligned} \tag{3.2.7}$$

The reason why these parameters may change a little can be explained when the physical aspects of this type of motor is studied in detail. As for now we assume that these parameters are subject to changes.

It is required that states x_2 and x_3 be estimated. Therefore designing a Luenberger observer for system (2.2.3) we have

$$\dot{z} = \begin{bmatrix} -k_1 & 1 \\ -(k_2+\alpha_2^o) & -\alpha_3^o \end{bmatrix} z + \begin{bmatrix} -k_1^2 + k_2 \\ -[(k_2+\alpha_2^o)k_1+\alpha_3^o k_2] \end{bmatrix} y + \begin{bmatrix} 0 \\ \beta_3^o \end{bmatrix} u \qquad (3.2.8)$$

The coefficients k_1 and k_2 are selected such that the eigenvalues of (3.2.8) are more negative than the eigenvalues of system (3.2.3).

The output of the observer is

$$\hat{x}_2 = z_1 + k_1 y \ , \quad \hat{x}_3 = z_2 + k_2 y \qquad (3.2.9)$$

With the help of the theory developed in chapter 2, the open-loop sensitivity model is given by

$$\dot{h} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\alpha_2^o & -\alpha_3^o \end{bmatrix} h - \beta_3^{o^{-1}} \hat{x}_2 \qquad (3.2.10)$$

The sensitivity functions can now be obtained from the model (3.2.10). The transformation matrices are given as

$$T = \beta_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad N_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.2.11)$$

As there are two estimated states and six parameters, there will be twelve sensitivity functions of the estimated states. They are as follows:

$$\frac{\delta \hat{x}_2}{\delta L_m} = - \frac{K_T^o R_m^o}{L_m^{o^3} J^o} h_{2,3} - \frac{1}{L_m} \hat{x}_2 = - 4h_{2,3} - \hat{x}_2 \qquad (3.2.12)$$

$$\frac{\delta \hat{x}_3}{\delta L_m} = - \frac{K_T^o R_m^o}{L_m^{o^2} J^o} h_{3,3} - \frac{1}{L_m} \hat{x}_3 = - 4h_{3,3} - \hat{x}_3 \qquad (3.2.13)$$

$$\frac{\delta \hat{x}_2}{\delta R_m} = \frac{K_T^o}{L_m^{o^2} J^o} h_{2,3} = 2\ h_{2,3} \qquad (3.2.14)$$

$$\frac{\delta \hat{x}_3}{\delta R_m} = \frac{K_T}{L_m^2 J^o} h_{3,3} = 2\ h_{3,3} \qquad (3.2.15)$$

$$\frac{\delta \hat{x}_2}{\delta J} = - \frac{K_T^o B}{J^{o^3} L_m^o} h_{2,3} - \frac{1}{J^o} \hat{x}_2 = - 1.78h_{2,3} - 0.89\hat{x}_2 \qquad (3.2.16)$$

$$\frac{\delta \hat{x}_3}{\delta J} = - \frac{K_T^o B}{J^{o^3} L_m^o} h_{3,3} - \frac{1}{J^o} \hat{x}_3 = - 1.78h_{3,3} - 0.89\hat{x}_3 \qquad (3.2.17)$$

$$\frac{\delta \hat{x}_2}{\delta B} = \frac{K_T^o}{L_m^o J^{o^2}} h_{2,3} = 1.78h_{2,3} \qquad (3.2.18)$$

$$\frac{\delta \hat{x}_3}{\delta B} = \frac{K_T^o}{L_m^o J^{o^2}} h_{3,3} = 1.78h_{3,3} \qquad (3.2.19)$$

$$\frac{\delta \hat{x}_2}{\delta K_T} = \frac{K_b^o K_T^o}{L_m^{o^2} J^{o^2}} h_{1,3} + \frac{1}{K_T^o} \hat{x}_2 = 1.78h_{1,3} + 0.4\hat{x}_2 \qquad (3.2.20)$$

$$\frac{\delta\hat{x}_3}{\delta K_T} = \frac{K_b^o K_T^o}{L_m^{o\,2} J^{o\,2}} h_{2,3} + \frac{1}{K_T^o}\hat{x}_3 = 1.78 h_{2,3} + 0.4\hat{x}_3 \tag{3.2.21}$$

$$\frac{\delta\hat{x}_2}{\delta K_b} = \left(\frac{K_T^o}{L_m^o J^o}\right)^2 h_{1,3} = 4\ h_{1,3} \tag{3.2.22}$$

$$\frac{\delta\hat{x}_3}{\delta K_b} = \left(\frac{K_T^o}{L_m^o J^o}\right)^2 h_{2,3} = 4\ h_{2,3} \tag{3.2.23}$$

Thus we see that, with the knowledge of only three sensitivity functions $h_{1,3}$, $h_{2,3}$, and $h_{3,3}$, we have generated all the twelve sensitivity functions, (3.2.12) to (3.2.23). Figure 3.1 is the diagram for generating the estimated state sensitivity functions for Example 1. The plots of the sensitivity functions (3.2.12) to (3.2.23) are illustrated in Figures 3.2 to 3.5.

Comments

While developing the sensitivity model, it is noticed that the partitioned form of (3.2.3) has A_{12} to be a zero matrix. This eliminates the need for the output sensitivity function $\frac{\delta x_1}{\delta g}$ and thus reduces the sensitivity model to a (n-r) dimensional model. Thus it can be concluded that if equation (2.2.7) has rank (n-r), then the sensitivity model for the observer will be a (n-r) dimensional model. If the rank is higher, then we must use the n-dimensional model as derived in Section 2.5.

It must be noted that the model generates the open-loop, estimated state sensitivity functions.

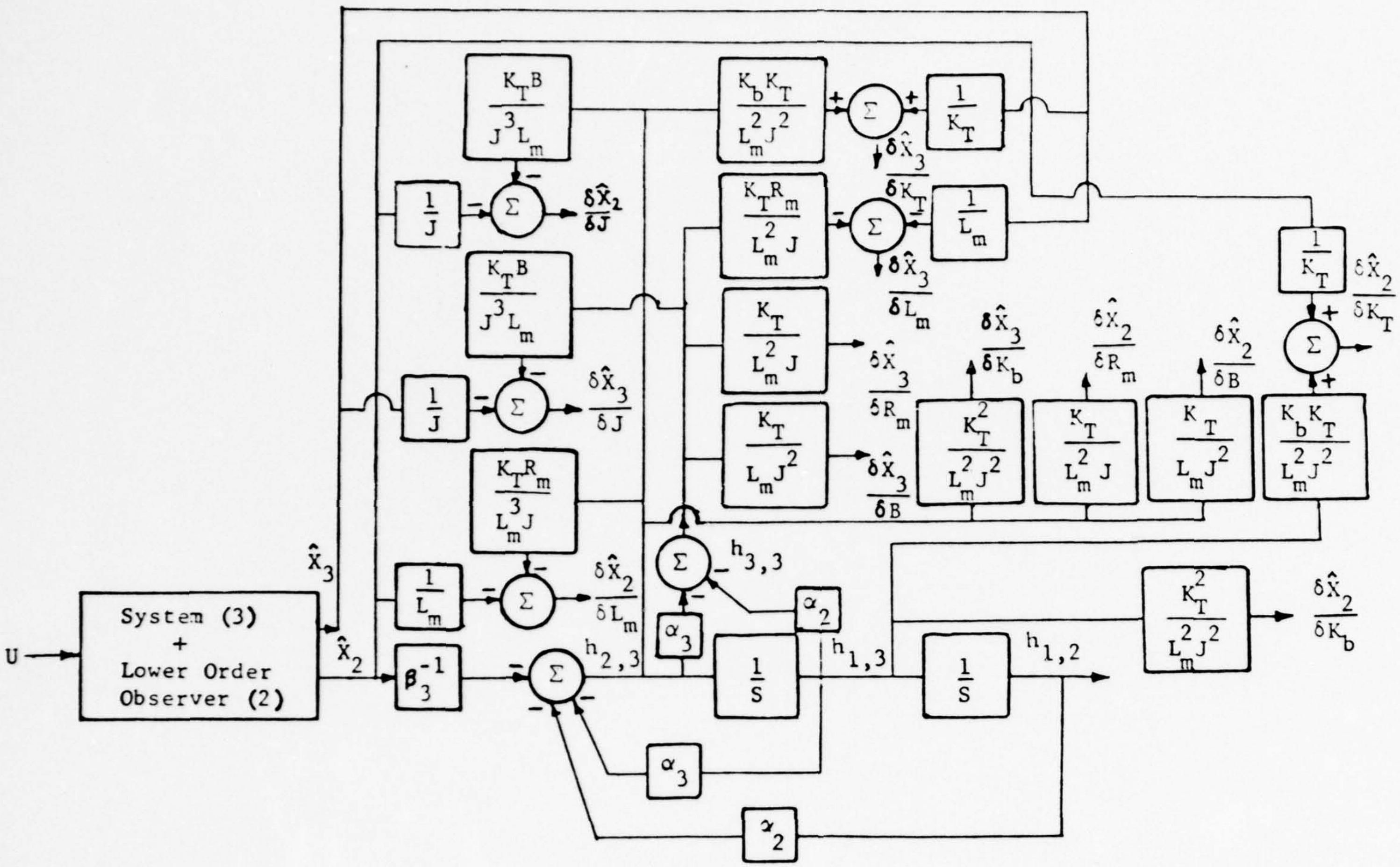

Figure 3.1 Diagram for generating the estimated state sensitivity functions for Example 1.

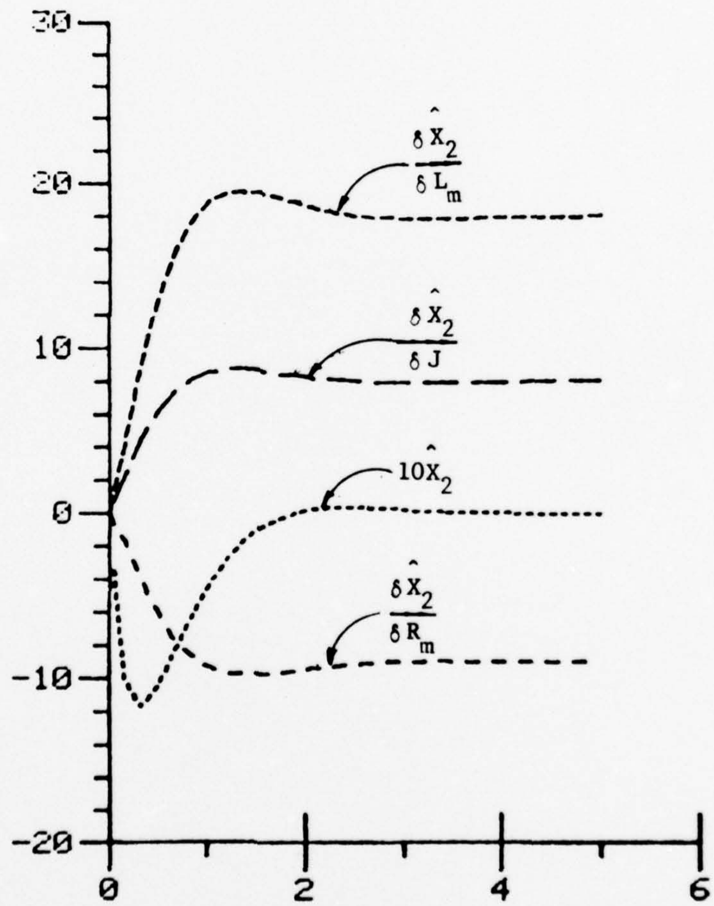

Figure 3.2 Estimated state $\hat{X}_2$ and the sensitivity functions for Example 1.

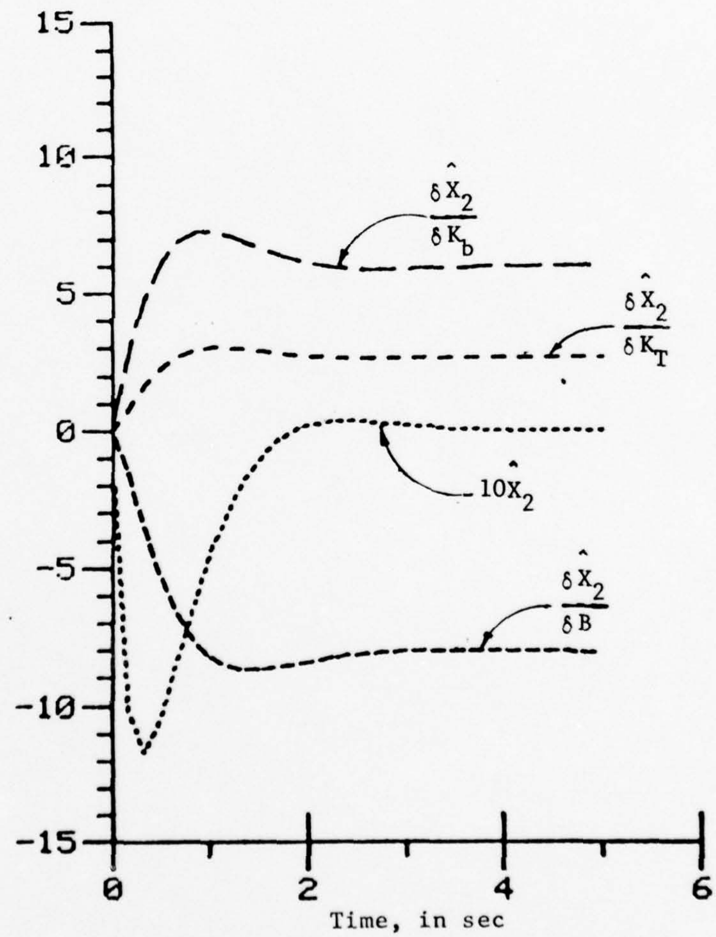

Figure 3.3 Estimated state $\hat{X}_2$ and the sensitivity functions for Example 1.

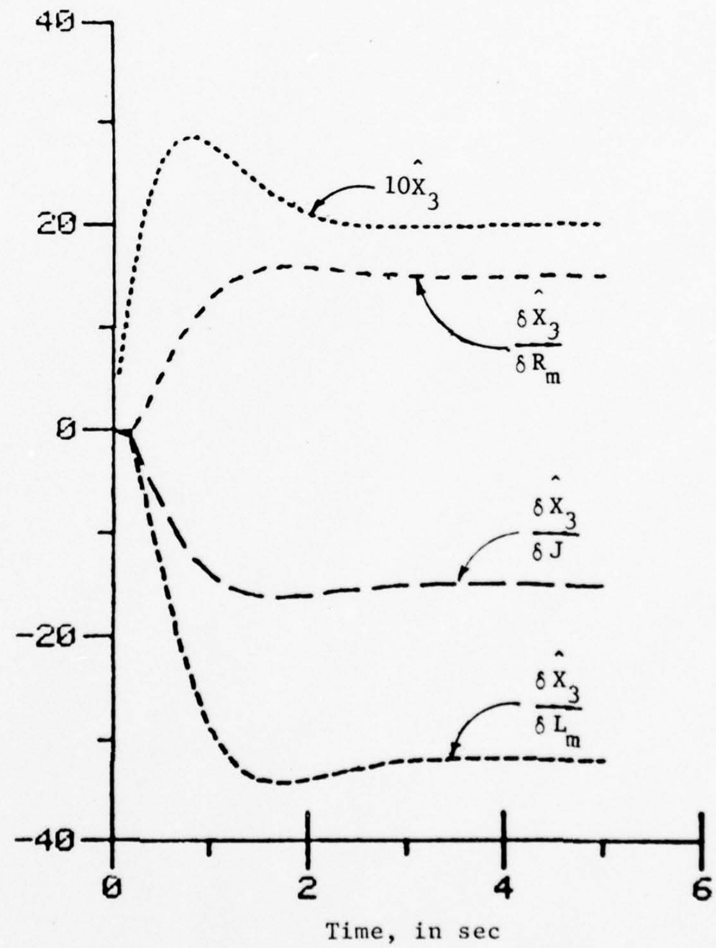

Figure 3.4 Estimated state $\widehat{X_3}$ and the sensitivity functions for Example 1.

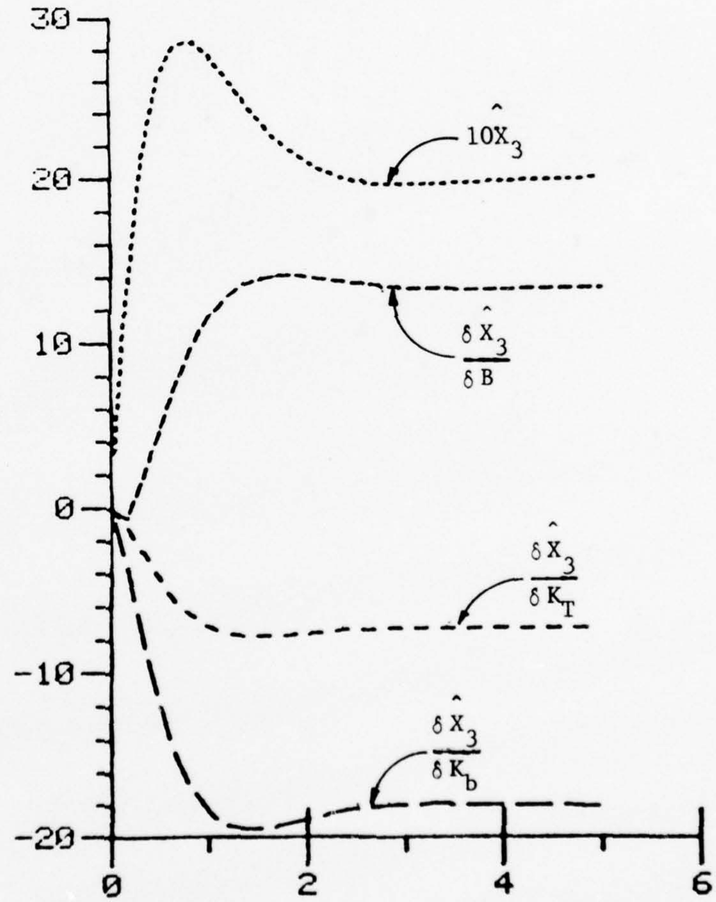

Figure 3.5 Estimated state $\hat{X}_3$ and the sensitivity functions for Example 1.

It is noticed from Figures 3.2 to 3.5 that all the sensitivity functions have greater magnitude than the actual estimated states $\hat{x}_2$ and $\hat{x}_3$. An example to what can happen is given as follows.

For a ten percent increase in parameters the deviations in $\hat{x}_2$ are tabulated below. These deviations are the steady state deviations and will effect the steady state response of $\hat{x}_2$.

$\delta\hat{x}_2$ will be		1.9	for a ten percent increase in			L_m
"	"	1.0125	"	"	"	J
"	"	-1.8	"	"	"	R_m
"	"	0.65	"	"	"	K_b
"	"	0.675	"	"	"	K_T
"	"	-0.84375	"	"	"	B

$\delta\hat{x}_2$ will be 1.583 as a combined effect.

Therefore limit $\hat{x}_2$ may always be in steady state. This could be a very bad disadvantage if $\hat{x}_2$ is expected to reach zero in finite time.

It is evident that the steady state error in $\hat{x}_2$ or $\hat{x}_3$ is directly proportional to the increase or decrease in parameters. At certain values of the parameters there may be no steady state error at all, or a small steady state error and for large deviations in parameters a large steady state error. The compounded effect may be disadvantageous.

In a physical system if the parameter deviations are small, 0.01 percent, and if small steady state error is tolerated, then the lower order observer will serve well. If not, then another lower order observer will

have to be designed. Note that even though the eigenvalues selected make the estimated states decay to zero in 2 seconds, the parameters may cause a big steady state error.

Therefore this observer may or most probably may not be used to estimate the two states $\hat{X}_2$ and $\hat{X}_3$.

3.3. Example No. 2

From the equations of the famous stick-cart system, the linearized stick part is given by

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= aX_1 + bu \end{aligned} \tag{3.3.1}$$

where, X_1 is linear displacement and X_2 velocity.

$$a = \frac{3g(m+M)}{L(m+4M)} \tag{3.3.2}$$

$$b = \frac{-3}{L(m+4M)} \tag{3.3.3}$$

where M is the mass of the cart

m is the mass of the stick

L is half the length of the stick

g is the acceleration due to gravity.

Typical values of these parameters are:

$$\begin{aligned} g^o &= 9.78 \text{ m/sec}^2 \\ M^o &= 10 \text{ Kg} \\ m^o &= 1.24 \text{ Kg} \\ L^o &= 1 \text{ meter.} \end{aligned} \tag{3.3.4}$$

Equation (3.3.1) can be put into the form

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} u\,, \quad y = X_1 \tag{3.3.5}$$

where

$$\alpha_1^o = -a = -\frac{3g^o(m^o + M^o)}{L^o(m^o + 4M)} = -8$$

$$\beta_2^o = b = -\frac{3}{L^o(m^o + 4M^o)} = -0.0727. \tag{3.3.6}$$

Thus the eigenvalue will be $\pm\sqrt{8}$.

In this system there are four parameters and they are given as follows.

$$\begin{aligned} g_1 &= g \\ g_2 &= M \\ g_3 &= m \\ g_4 &= L \end{aligned} \tag{3.3.7}$$

All these parameters are subjected to slight changes and thus could affect the estimated states.

It is required that X_2 be estimated.

The Luenberger observer for system (3.3.1) is given by

$$\dot{Z} = [-k]Z + [-k^2 + a^o]y + [\beta_2^o]u \tag{3.3.8}$$

$$\hat{X}_2 = Z + ky. \tag{3.3.9}$$

The coefficient k is selected to be more negative than the most negative eigenvalue of (3.3.1).

$$k > \sqrt{8} \tag{3.3.10}$$

Using the same techniques as adopted in Chapter 2, we design the open-loop sensitivity model. The model is given by (3.3.11)

$$h^o = \begin{bmatrix} 0 & 1 \\ -\alpha_1^o & 0 \end{bmatrix} h - \beta_2^{o^{-1}} X_1. \tag{3.3.11}$$

The transformation matrix T and N_1 are given as

$$T = \beta_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_1 = [0 \quad 1]. \tag{3.3.12}$$

As there are four parameters and one estimated state, there will be four sensitivity functions. These sensitivity functions are given below, and they can be generated from the sensitivity model (3.3.11).

$$\frac{\delta \hat{X}_2}{\delta g} = \left(\frac{3}{L^o(m^o + 4M^o)}\right)^2 (m^o + M^o) h_{1,2} = 0.06 h_{1,2} \tag{3.3.13}$$

$$\frac{\delta \hat{X}_2}{\delta M} = -\frac{27 g^o m^o}{L^{o2}(m^o + 4M^o)^3} h_{1,2} - \frac{4}{m^o + 4M^o} \hat{X}_2 \tag{3.3.14}$$

$$= -0.0047 h_{1,2} - 0.097 \hat{X}_2$$

$$\frac{\delta \hat{X}_2}{\delta m} = \frac{27 g^o M^o}{L^{o2}(m^o + 4M^o)^3} h_{1,2} - \frac{1}{m^o + 4M^o} \hat{X}_2 \tag{3.3.15}$$

$$= 0.038 h_{1,2} - 0.024 \hat{X}_2$$

$$\frac{\delta \hat{X}_2}{\delta L} = -\frac{9 g^o (m^o + M^o)}{L^{o3}(m^o + 4M^o)^2} h_{1,2} - \frac{1}{L^o} \hat{X}_2 \tag{3.3.16}$$

$$= -0.6 h_{1,2} - \hat{X}_2$$

We only require $h_{1,2}$ out of the three known sensitivity functions to generate the sensitivity functions for the four parameters in this problem.

Figure 3.6 is the diagram for generating the estimated states sensitivity functions for Example 2. The plots of the sensitivity functions (3.3.13) to (3.3.16) are illustrated in Figures 3.7 and 3.8.

Comments

From Figures 3.7 and 3.8 it is noticed that the sensitivity is in the order of 10^{11} and therefore the observer of Example 2 is highly sensitive to the parameters of the system. This is a very good example for the type of systems for which lower order observers cannot be designed. Thus the observer of Example 2 is useless, as it will make tremendous errors in the estimation.

3.4. Example No. 3

The equations of a mercury thermometer from [6] are given by the equation

$$\left[C_g C_m D^2 + \left(\frac{C_g}{R_m} + \frac{C_m}{R_g} + \frac{C_m}{R_m}\right)D + \frac{1}{R_g R_m}\right]\theta_m = \frac{1}{R_m R_g}\theta_o \qquad (3.4.1)$$

where

R_g is the resistance of the glass

C_g is the capacitance of the glass

R_m is the resistance of mercury

C_m is the capacitance of the mercury

θ_m is the temperature of the mercury

θ_o is the temperature of the environment.

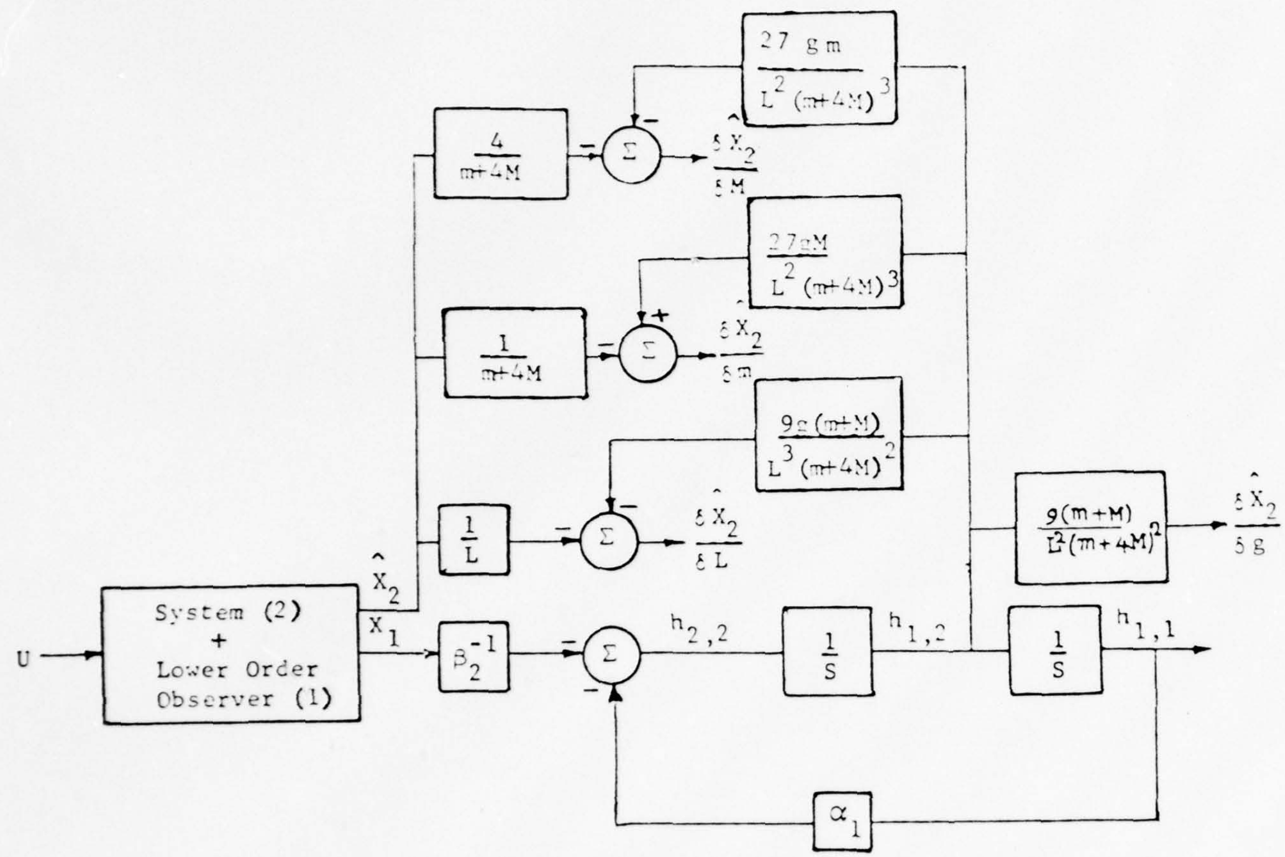

Figure 3.6 Diagram for generating the estimated state sensitivity functions for Example 2.

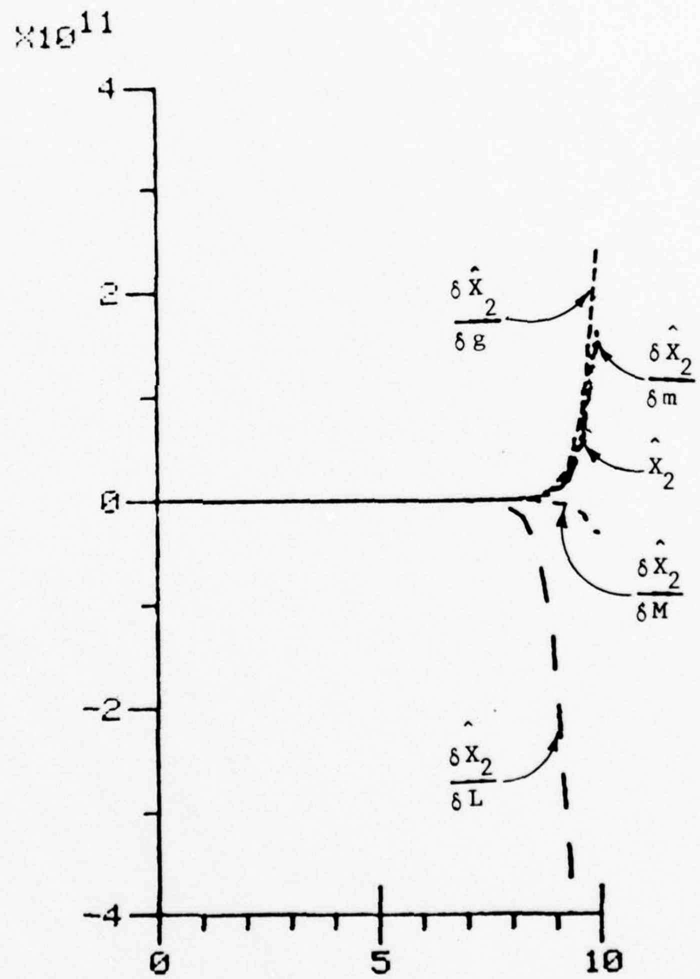

Figure 3.7 Estimated state $\hat{x}_2$ and the sensitivity functions for Example 2.

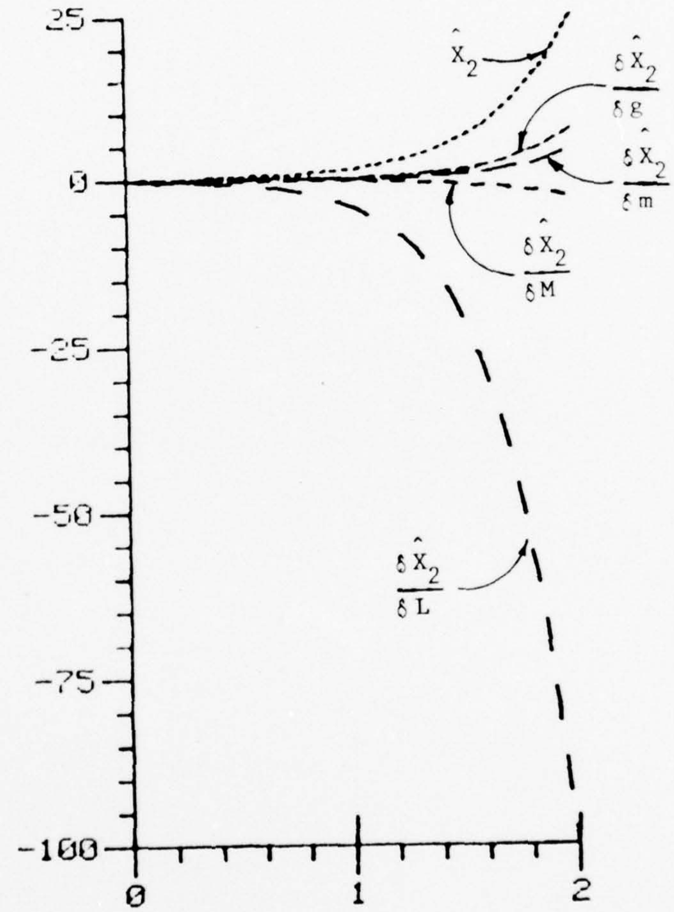

Figure 3.8 Estimated state $\hat{x}_2$ and the sensitivity functions for Example 2.

Typical values of the parameters for 10 lbs. of glass and 100 lbs. of mercury

$$\begin{aligned} R_g^o &= 1/5 \text{ Degree/(Btu/minute)} \\ C_g^o &= 2 \quad \text{Btu/degree} \\ R_m^o &= 1/60 \text{ Degree/(Btu/minute)} \\ C_m^o &= 3 \quad \text{Btu/degree.} \end{aligned} \tag{3.4.2}$$

Equation (3.4.1) can be put into the form

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix} X + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} u, \quad y = x_1 \tag{3.4.3}$$

where

$$\alpha_1^o = \frac{1}{C_m^o R_m^o C_g^o R_g^o} = 50 \tag{3.4.4}$$

$$\alpha_2^o = \left(\frac{1}{C_m^o R_m^o} + \frac{1}{C_g^o R_g^o} + \frac{1}{C_g^o R_m^o}\right) = 52.5 \tag{3.4.5}$$

$$\beta_2^o = \frac{1}{C_m^o R_m^o C_g^o R_g^o} = 50. \tag{3.4.6}$$

There are four parameters in this system. They are

$$\begin{aligned} g_1 &= C_m \\ g_2 &= R_m \\ g_3 &= C_g \\ g_4 &= R_g. \end{aligned} \tag{3.4.7}$$

All these parameters are affected by the conditions of temperature and physical materials, therefore they tend to change. A detailed study on the topic of heat transfer will verify this assumption.

It is required that state X_2 be estimated. Note that X_2 represents the temperature at the inner surface between the glass case and mercury. The Luenberger observer for system (3.4.1) is given as

$$\dot{Z} = [-(k+\alpha_2^o)]Z + [-(k^2+\alpha_2^o k+\alpha_1^o)]y + [\beta_2^o]u \tag{3.4.8}$$

$$\hat{X}_2 = Z + ky \tag{3.4.9}$$

$(\alpha_2 + k)$ must be more negative than -50.

The sensitivity model for the open-loop sensitivity functions is given by

$$h^o = \begin{bmatrix} 0 & 1 \\ -\alpha_1^o & -\alpha_2^o \end{bmatrix} h - \beta_2^{o^{-1}} X_1 \tag{3.4.10}$$

The transformation matrices are given as

$$T = \beta_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N = [0 \quad 1]. \tag{3.4.11}$$

As there are four parameters and one estimated state, there will be four sensitivity functions which can be written as

$$\frac{\delta \hat{X}_2}{\delta C_m} = -\frac{\beta_2^{o^2}}{C_m^o} h_{1,2} - \frac{1}{C_m^o}\hat{X}_2 \tag{3.4.12}$$

$$= -833.33h_{1,2} - 0.33\hat{X}_2$$

$$\frac{\delta\hat{X}_2}{\delta R_m} = -\frac{\beta_2^{o^2}}{R_m^o} h_{1,2} - \frac{1}{R_m^o}\hat{X}_2 \qquad (3.4.13)$$

$$= -15\times 10^4 h_{1,2} - 60\hat{X}_2$$

$$\frac{\delta\hat{X}_2}{\delta C_g} = -\frac{\beta_2^{o^2}}{C_g^o} h_{1,2} - \frac{1}{C_g^o}\hat{X}_2 \qquad (3.4.14)$$

$$= -1250 h_{1,2} - 0.5\hat{X}_2$$

$$\frac{\delta\hat{X}_2}{\delta R_g} = -\frac{\beta_2^{o^2}}{R_g^o} h_{1,2} - \frac{1}{R_g^o}\hat{X}_2 \qquad (3.4.15)$$

$$= -12{\cdot}5\times 10^3 h_{1,2} - 5\hat{X}_2.$$

These are the four sensitivity functions that are generated from the sensitivity model (3.4.10). Figure 3.9 is the diagram for generating the estimated states sensitivity functions for Example 3. The plots of these sensitivity functions, (3.4.12) to (3.4.15) are illustrated in Figures 3.10 and 3.11.

Comments

From Figures 3.10 and 3.11 we notice that all the sensitivity functions, except $\frac{\delta\hat{X}_2}{\delta R_m}$, decay to zero by three seconds. The parameter R_m, or the resistance of mercury causes a steady state error to exist in the temperature $\hat{X}_2$ or the temperature at the inner surface between the glass case and mercury. A ten percent increase in the resistance will cause a 1.66×10^{-3} deviation in the temperature $\hat{X}_2$. As the temperature itself reaches a steady state of -30×10^{-3} the positive deviation is not considerable.

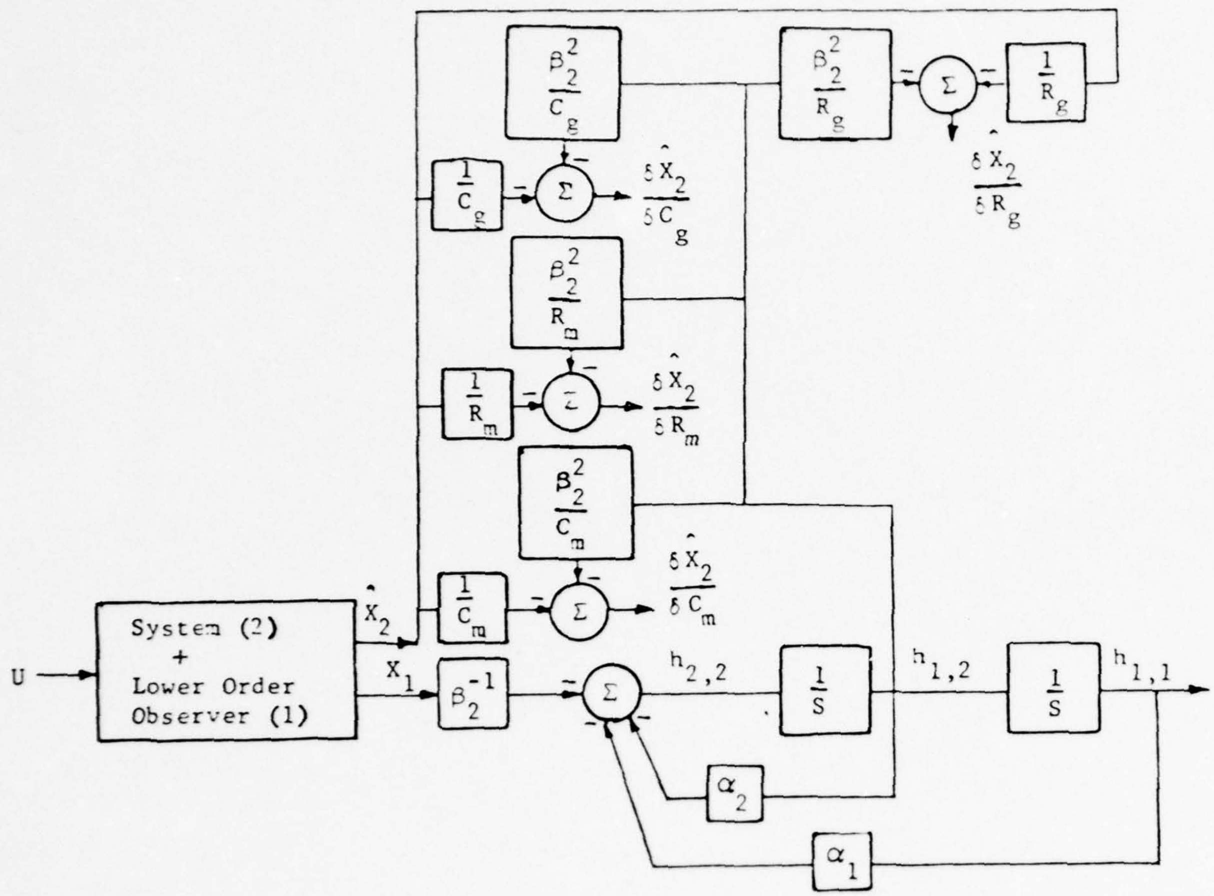

Figure 3.9 Diagram for generating the estimated state sensitivity functions for Example 3.

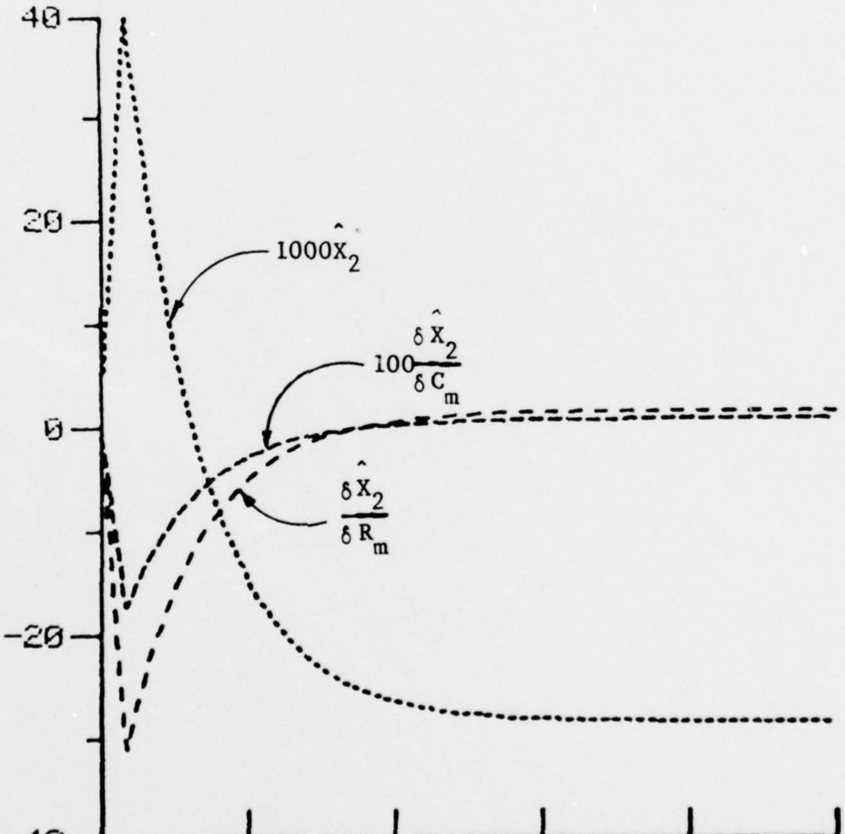

Figure 3.10 Estimated state $\hat{x}_2$ and the sensitivity functions for Example 3.

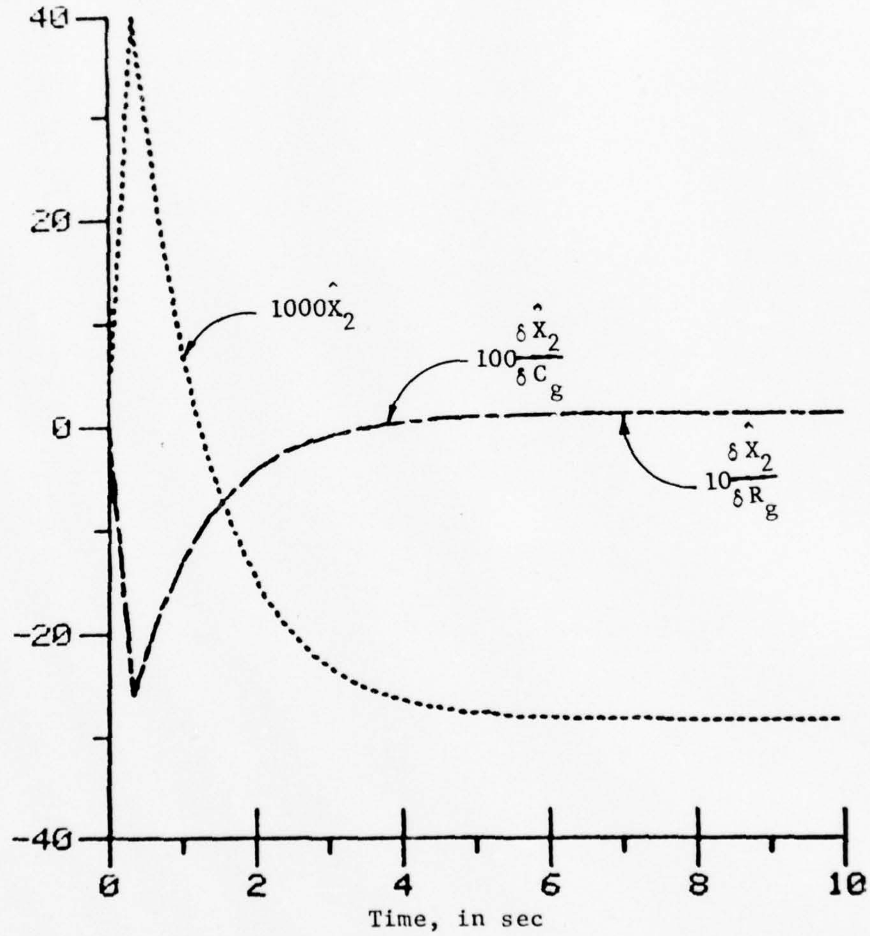

Figure 3.11 Estimated state $\hat{x}_2$ and the sensitivity functions for Example 3.

It is noticed that the deviations of the capacitances C_m and C_g are of the magnitude 10^{-3}.

The deviation due to R_g is also of the magnitude 10^{-3}. The maximum or minimum deviations are negligible as compared to the maximum and minimum values of the temperature $\hat{X}_2$. It must be mentioned though that the resistance of mercury R_m causes a slight steady state error or deviation in steady state.

Thus it is seen that the lower order observer for Example 3 is reasonably independent of variations in parameters and thus is a good open-loop estimator.

3.5. Example No. 4

A system is described by the equations

$$\begin{aligned} \dot{X}_1 &= a_1 X_2 \\ \dot{X}_2 &= a_2 X_1 + a_3 X_2 + bu \end{aligned} \tag{3.5.1}$$

$$y = X_1 \tag{3.5.2}$$

where a_1, a_2, a_3, and b are assumed to be parameters that change very slightly. The nominal value of these parameters are

$$a_1^o = 2,\ a_2^o = -2,\ a_3^o = -3,\ b^o = 2. \tag{3.5.3}$$

There are four parameters and they are given by

$$\begin{aligned} g_1 &= a_1 \\ g_2 &= a_2 \\ g_3 &= a_3 \\ g_4 &= b. \end{aligned} \tag{3.5.4}$$

It is required to estimate the state X_2. The Luenberger observer for system (3.5.1) is given by

$$\dot{Z} = [a_3 - ka_1]Z + [-(k^2 a_1 - a_3 k - a_2)]y + [b]u \qquad (3.5.5)$$

$$\hat{X}_2 = Z + ky. \qquad (3.5.6)$$

The eigenvalues of $(a_3 - ka_1)$ must be more negative than the eigenvalues of (3.5.1), that are $-\frac{3}{2} \pm j\frac{\sqrt{7}}{2}$.

The sensitivity model for the open-loop sensitivity functions is given by

$$h^o = \begin{bmatrix} 0 & 1 \\ -\alpha_1^o & -\alpha_2^o \end{bmatrix} h - \frac{1}{ba_1} X_1 \qquad (3.5.7)$$

where

$$\alpha_1^o = -a_1^o a_2^o = 4 \qquad (3.5.8)$$

$$\alpha_2^o = -a_3^o = 3.$$

The transformation matrices are given as

$$T = b\begin{bmatrix} a_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N = [0 \quad 1]. \qquad (3.5.9)$$

The sensitivity functions are now generated from the sensitivity model (3.5.7) and are given by

$$\frac{\delta \hat{X}_2}{\delta a_1} = -ba_2^o\, h_{1,2} = 4\, h_{1,2} \qquad (3.5.10)$$

$$\frac{\delta \hat{X}_2}{\delta a_2} = -ba_1^o\, h_{1,2} = -4\, h_{1,2} \qquad (3.5.11)$$

$$\frac{\delta \hat{X}_2}{\delta a_3} = -b\ h_{2,2} = -2\ h_{2,2} \tag{3.5.12}$$

$$\frac{\delta \hat{X}_2}{\delta b} = \frac{1}{b}\ \hat{X}_2 = 0.5\ \hat{X}_2. \tag{3.5.13}$$

The plots of these sensitivity functions are given in Figures 3.13 and 3.14. Figure 3.12 is the diagram for generating the sensitivity functions for Example 4.

Comments

Of the four examples, this example has the best lower order observer, as far as steady state error goes. The deviations will be of the order of 10^{-2}. This can be seen in Figures 3.13 and 3.14. The sensitivity functions reach steady state of zero at an average of 6 sec. The influence that the different parameters have on the estimated state $\hat{X}_2$ can be easily seen in the figure. The four parameters have a different effect on $\hat{X}_2$ for example, a_1 has an opposite effect as does a_2. They both cancel each others effect on $\hat{X}_2$. Thus the parameters a_3 and b have a greater influence, and this is visible in Figure 3.14. Both parameters a_3 and b have an effect on the overshoot, undershoot and the steady state.

Thus it can be concluded that the lower order observer of Example 4 is quite independent of parameter variations and can thus be safely called a good observer or an estimator.

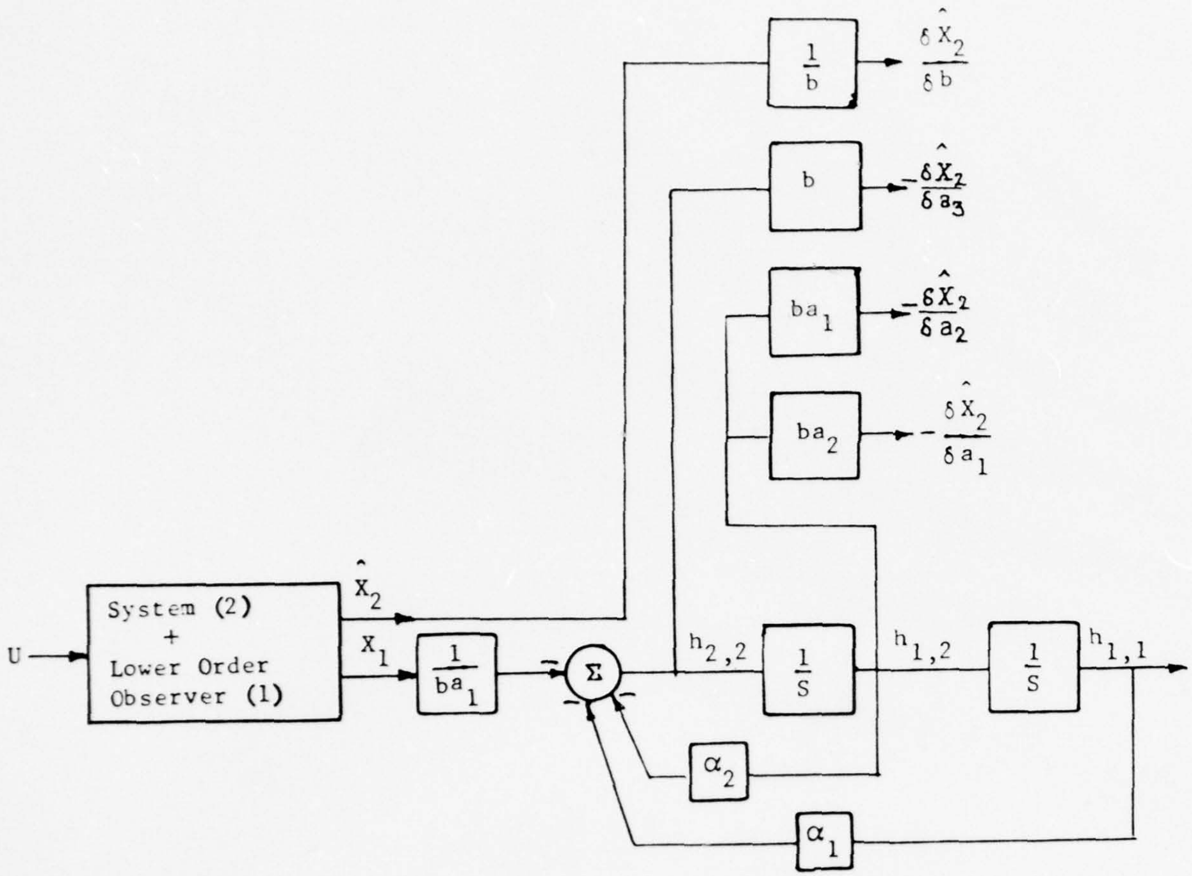

Figure 3.12 Diagram for generating the **estimated** state sensitivity functions for Example 4.

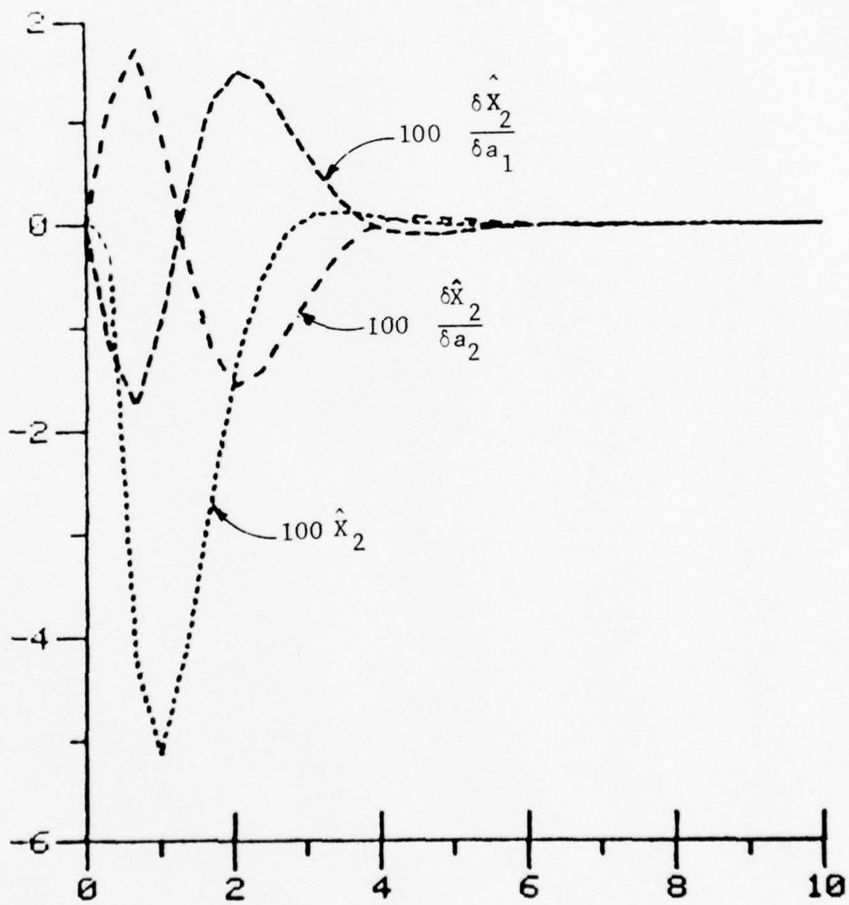

Figure 3.13 Estimated state $\hat{x}_2$ and the sensitivity functions for Example 4.

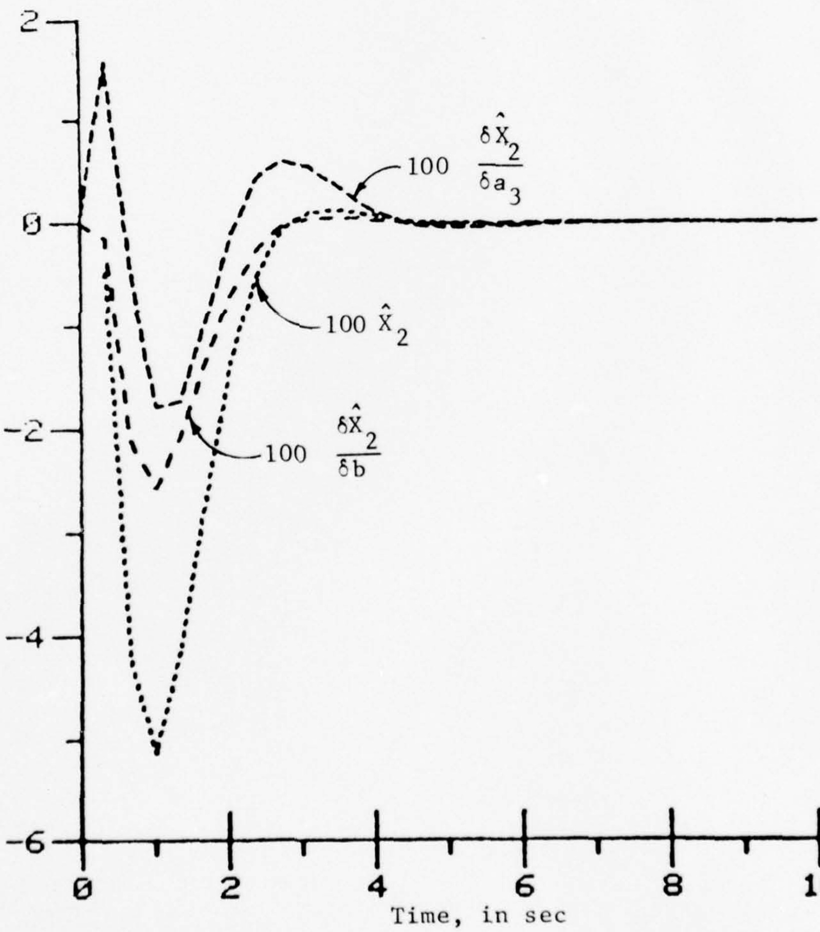

Figure 3.14 Estimated state $\hat{x}_2$ and the sensitivity functions for Example 4.

3.6. Analysis of the Sensitivity of Lower Order Observers in Feedback Loops

In the introduction to this chapter we have outlined the various points to be investigated. Some of these points fall into the feedback category and will be analyzed here in detail. In the case of observers in feedback we will study three main points of interest. They are: general sensitivity or effect of variation of parameters on closed-loop estimated states, effect of eigenvalues of an observer on sensitivity and finally comparison between the lower order observer and the full order observer in sensitivity. To accomplish this task an experimental analysis will be undertaken. Examples 5, 6, and 7 will be analyzed at the end of this section.

3.6.1. Example No. 5

A three-parameter linear time-invariant system is given by

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\alpha_2 & -\alpha_3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \beta_3 \end{bmatrix} u, \quad y = X_1. \tag{3.6.12}$$

The nominal values of these parameters are $\alpha_2^o = 3$, $\alpha_3^o = 4$, and $\beta_3^o = 2$. Using feedback the eigenvalues must be placed at (-1,-3,-4).

A lower order observer and a full order observer is required to be designed. The states X_2 and X_3 have to be estimated. The plots of the deviation in the estimated states to a ten percent increase in parameter value must be made.

In order to determine the effects of change of eigenvalues on the sensitivity of the lower order observer, at least three different eigenvalues must be selected.

Results: The plots of these deviations for eigenvalues, (-8,-10), (-11,-13), and (-15,-20) are in Figure 3.15 to Figure 3.20. The full order observers estimated states deviations are plotted in Figures 3.21 and 3.22.

The maximum and minimum deviations in the estimated states $\hat{X}_2$ and $\hat{X}_3$ to a ten percent increase in the parameters are tabulated in Table 3.1.

3.6.2. Example No. 6

A two parameter system is given by the equations

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} u, \quad y = X_1 \tag{3.6.13}$$

where the nominal values of the parameters are

$$\alpha_1^o = -8 \text{ and } \beta_2^o = 2.$$

It is required to estimate X_2. The feedback must be designed to adjust the eigenvalues of the system at (-3,-4). A similar procedure as adopted in Example 5, is followed and the following results are achieved.

Results: The plots for the deviations in $\hat{X}_2$ were recorded for three sets of observer eigenvalues which were (-10), (-20), and (-40). The plots for the full order observer was also recorded. These plots are available from Figure 3.23 to Figure 3.26. The maximum and minimum deviations to a ten percent increase in parameters were tabulated in Table 3.2.

3.6.3. Example No. 7

A linear time-invariant system is given by the equations

$$\dot{X} = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix} X + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} u, \quad y = X. \tag{3.6.14}$$

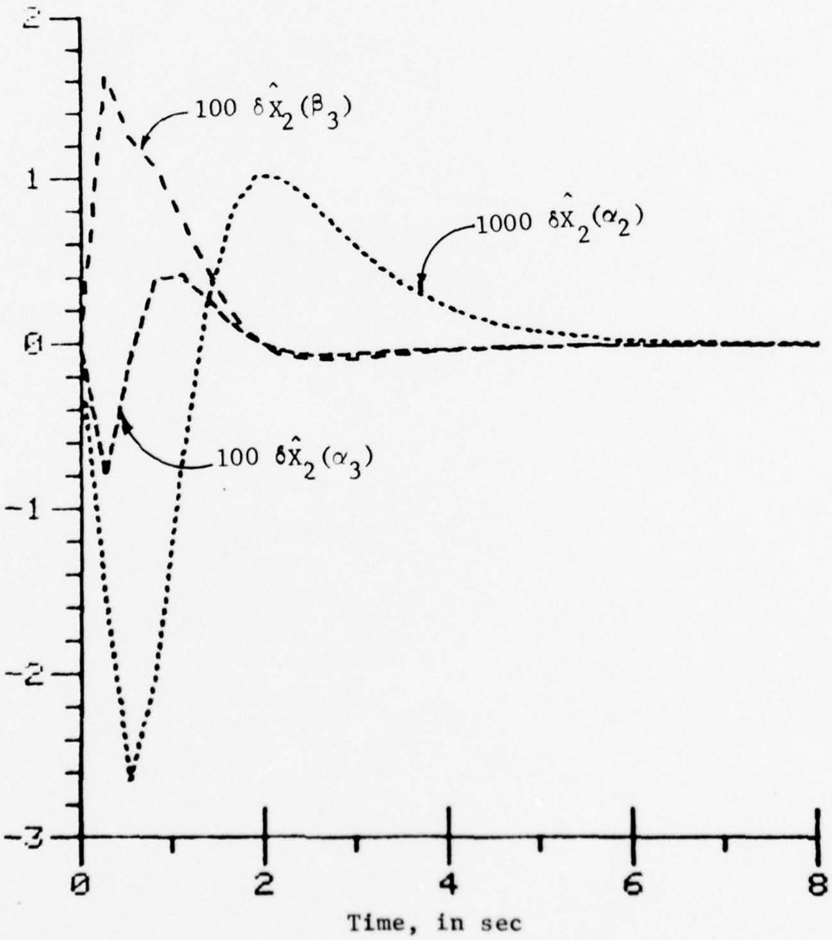

Figure 3.15 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 5. Observer eigenvalues at (-8, -10).

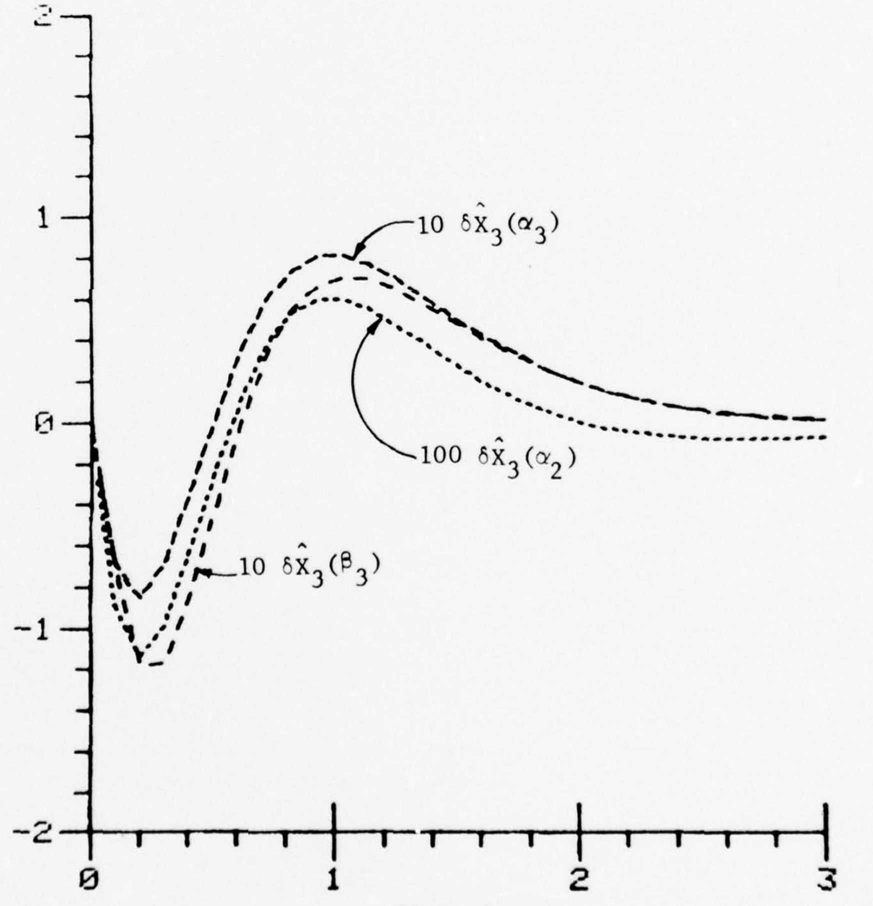

Figure 3.16 Estimated state $\hat{X}_3$ deviation for a ten percent increase in parameters, for Example 5. Observer eigenvalues at (-8, -10).

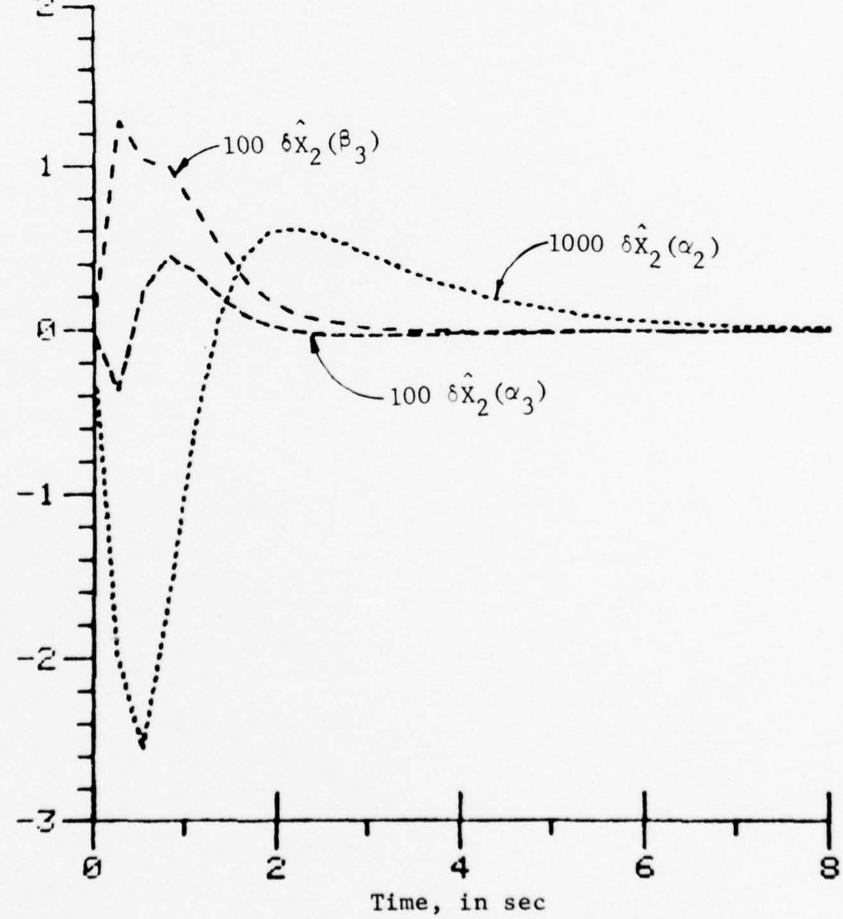

Figure 3.17 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 5. Observer eigenvalues at (-11, -13).

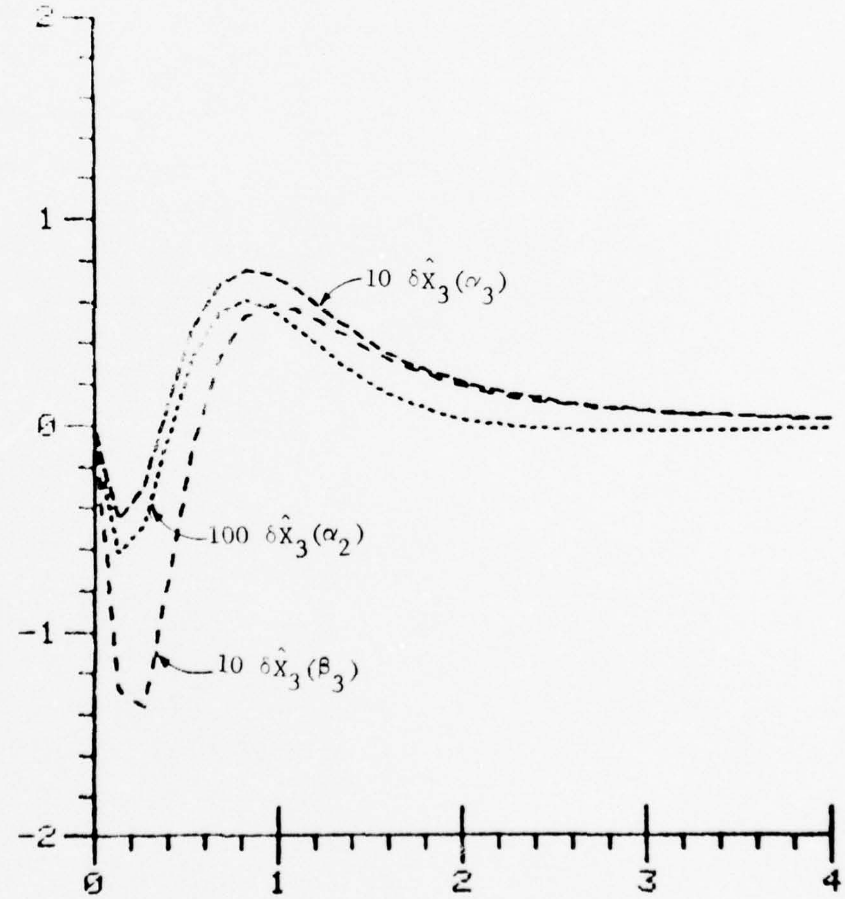

Figure 3.18 Estimated state $\hat{X}_3$ deviation for a ten percent increase in parameters, for Example 5. Observer eigenvalues at (-11, -13).

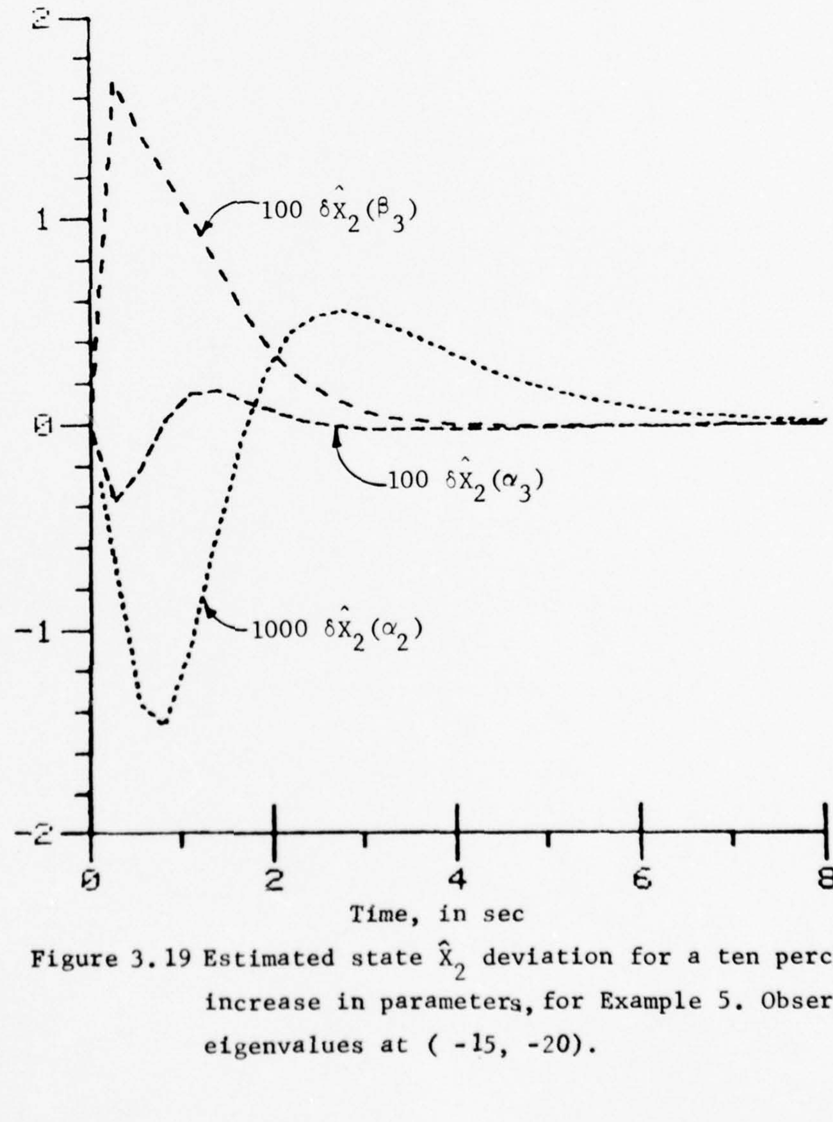

Figure 3.19 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 5. Observer eigenvalues at (-15, -20).

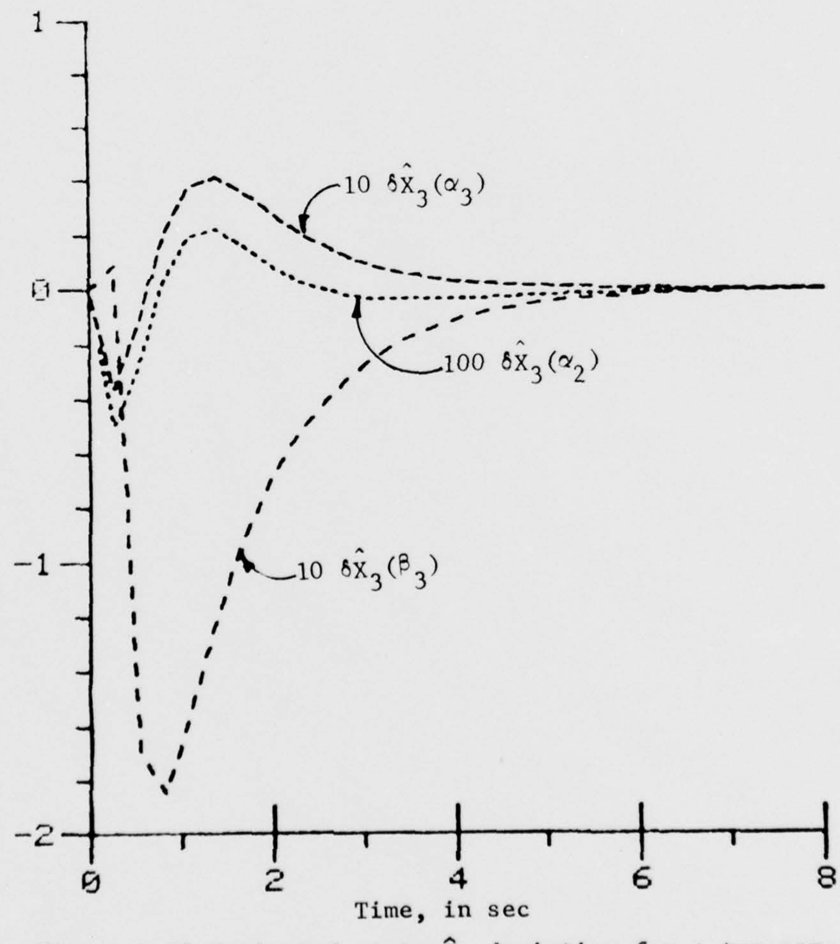

Figure 3.20 Estimated state $\hat{X}_3$ deviation for a ten percent increase in parameters, for Example 5. Observer eigenvalues at (-15,-20).

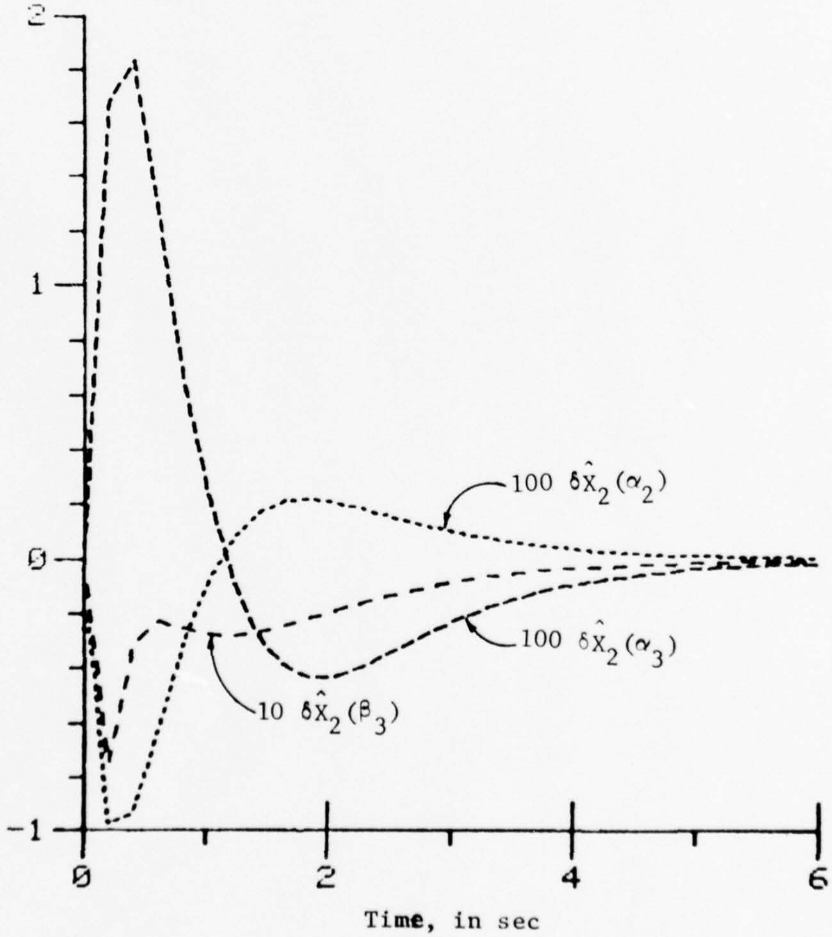

Figure 3.21 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parametes, for a Full Order Observer of Example 5.

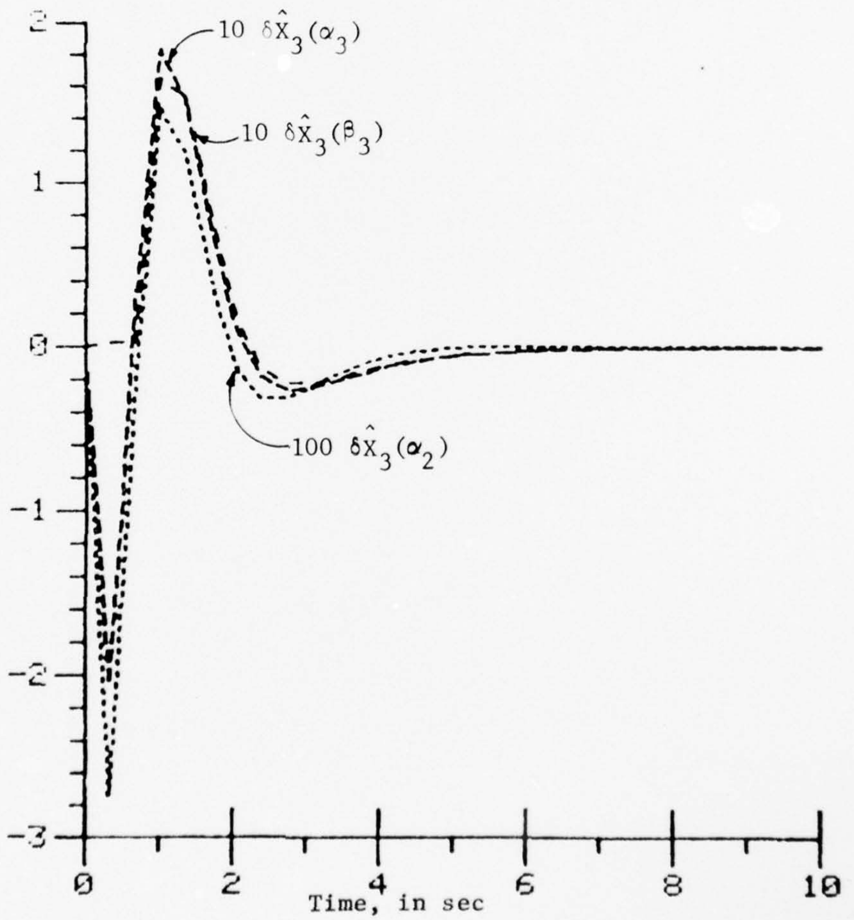

Figure 3.22 Estimated state $\hat{X}_3$ deviation for a ten percent increase in parameters, for the Full Order Observer of Example 5.

Table 3.1

For Example 5

Max. & Min. Increment $\delta\hat{X}$ for a 10% Increase in $\alpha_2, \alpha_3, \beta_3$

		Lower Order Observer Eigenvalues at (-8,-10)	Lower Order Observer Eigenvalues at (-11,-13)	Lower Order Observer Eigenvalues at (-15,-20)	Full Order Eigenvalues at (-8,-8,-10)
$\delta\hat{X}_3(\alpha_2)$	max	0.6×10^{-2}	0.6×10^{-2}	0.2×10^{-2}	1.6×10^{-2}
	min	-1.1×10^{-2}	-0.6×10^{-2}	-0.5×10^{-2}	-1.3×10^{-2}
$\delta\hat{X}_3(\alpha_3)$	max	0.8×10^{-1}	0.75×10^{-1}	0.4×10^{-1}	2.0×10^{-1}
	min	-0.85×10^{-1}	-0.46×10^{-1}	-0.36×10^{-1}	-0.9×10^{-1}
$\delta\hat{X}_3(\beta_3)$	max	0.7×10^{-1}	0.6×10^{-1}	0.1×10^{-1}	2.0×10^{-1}
	min	-1.2×10^{-1}	-1.39×10^{-1}	-1.87×10^{-1}	-0.6×10^{-1}
$\delta\hat{X}_2(\alpha_2)$	max	1.3×10^{-3}	0.65×10^{-3}	0.56×10^{-3}	0.4×10^{-2}
	min	-2.7×10^{-3}	-2.6×10^{-3}	-1.48×10^{-3}	-0.8×10^{-2}
$\delta\hat{X}_2(\alpha_3)$	max	0.4×10^{-2}	0.40×10^{-2}	0.18×10^{-2}	1.6×10^{-2}
	min	-0.8×10^{-2}	-0.5×10^{-2}	-0.38×10^{-2}	-0.7×10^{-2}
$\delta\hat{X}_2(\beta_3)$	max	1.61×10^{-2}	1.8×10^{-2}	1.68×10^{-2}	0.42×10^{-1}
	min	-0.1×10^{-2}	0.0	0.0	-1.2×10^{-1}

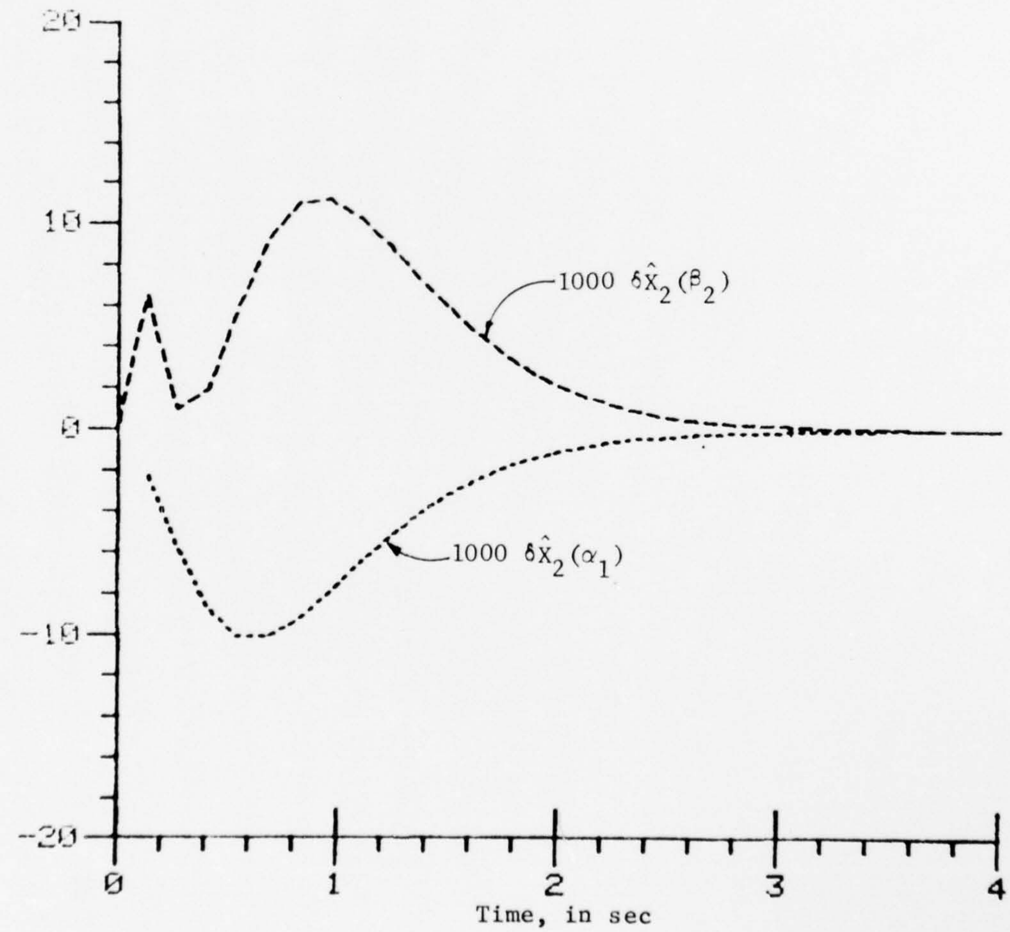

Figure 3.23 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 6. Observer eigenvalue at (-10).

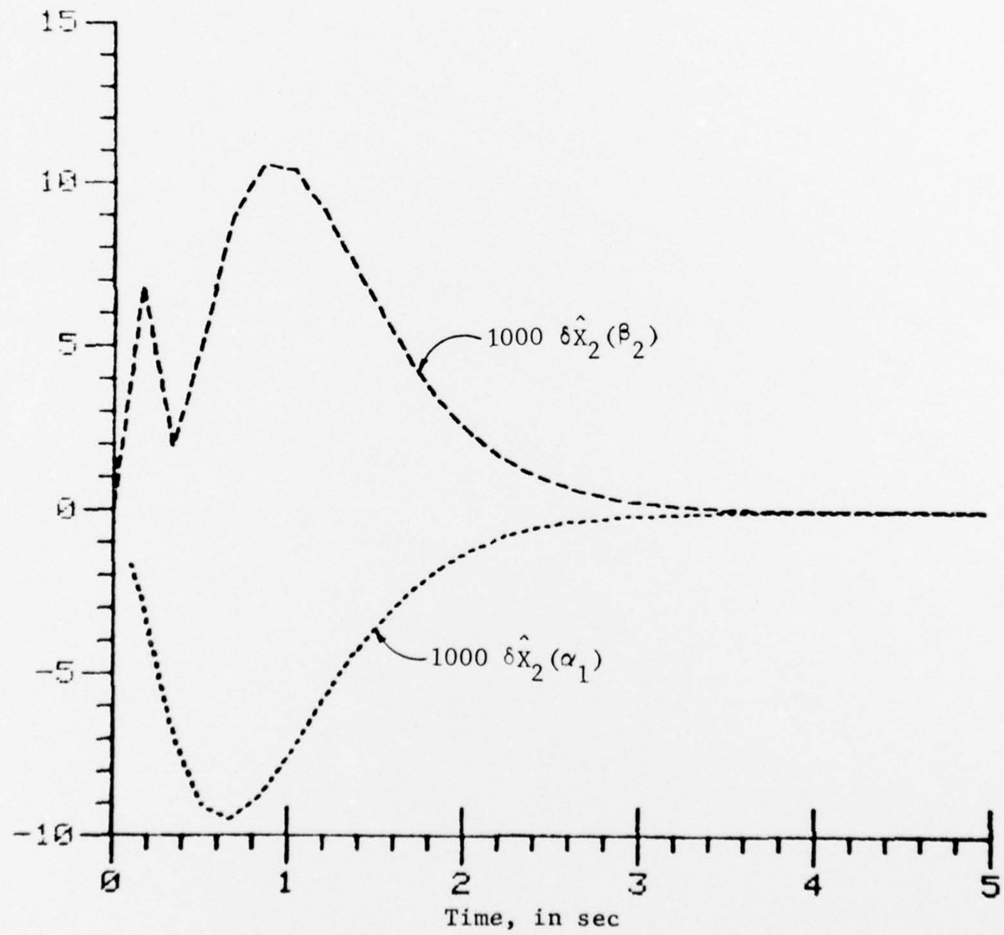

Figure 3.24 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 6. Observer eigenvalue at (-20).

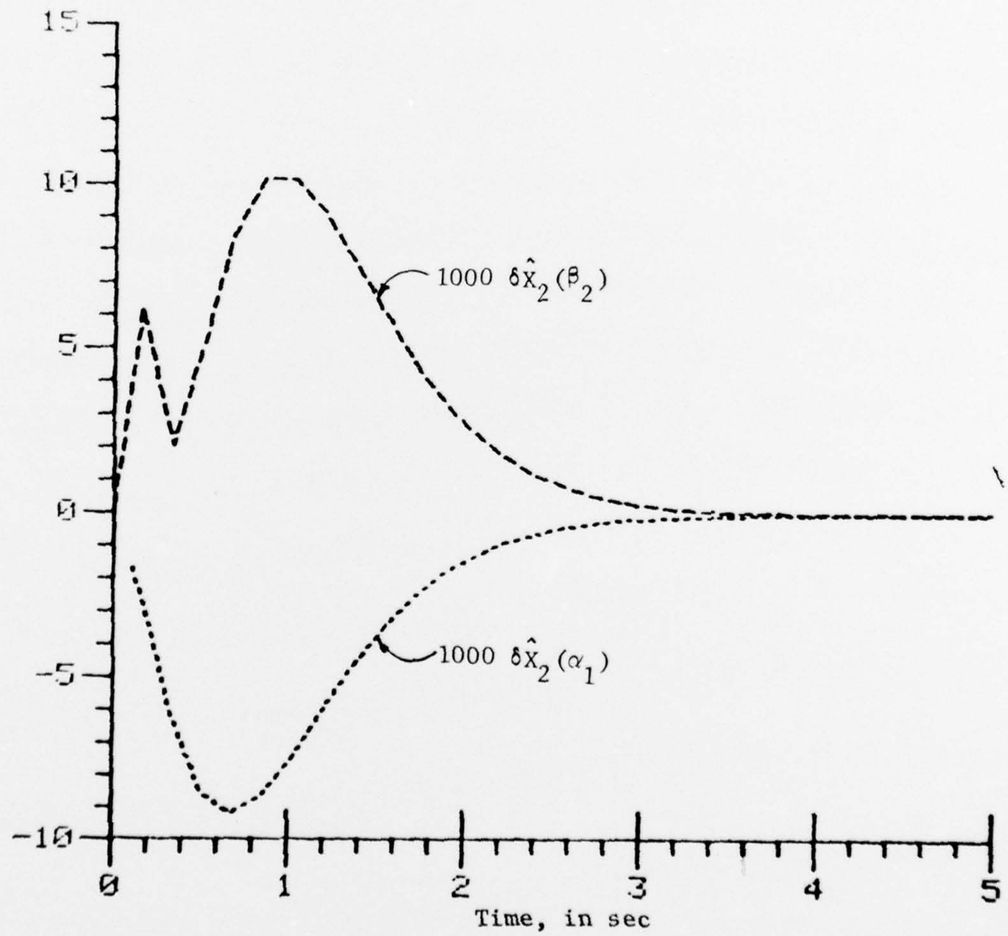

Figure 3.25 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 6. Observer eigenvalue at (-40).

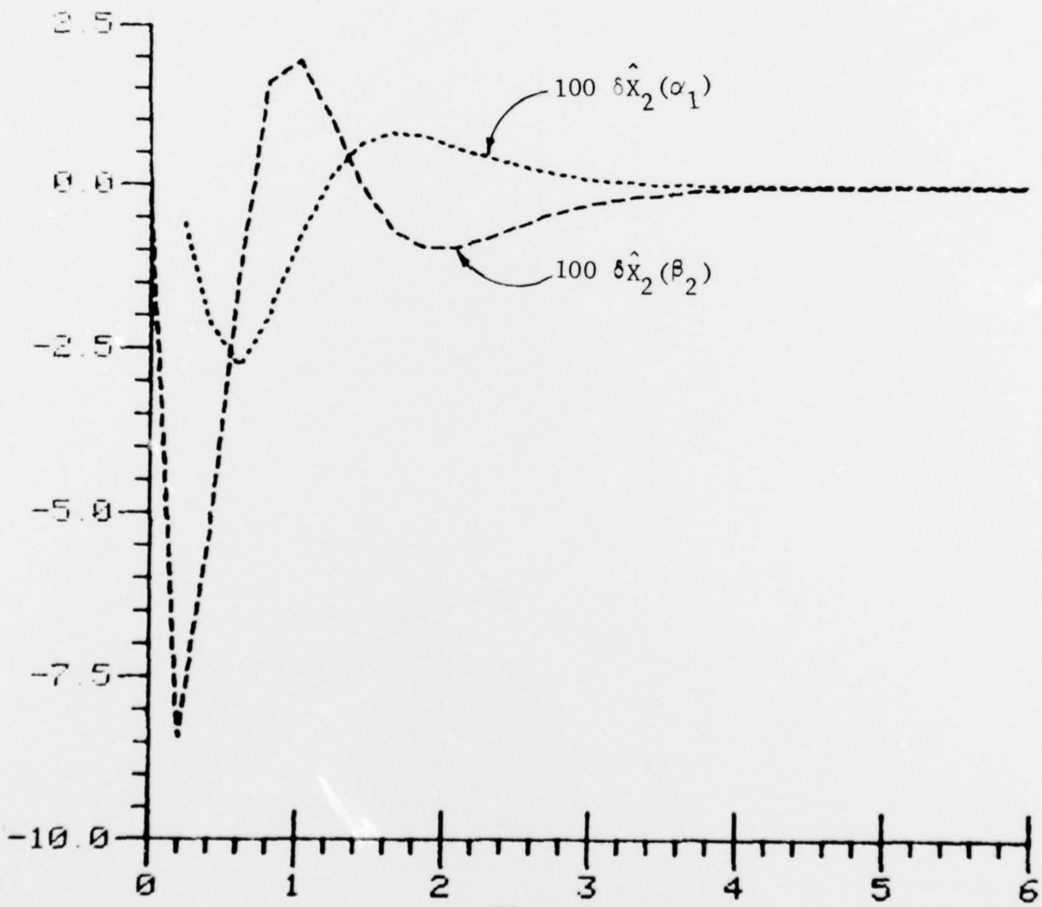

Figure 3.26 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 6. Full Order Observer.

Table 3.2

For Example 6

Max. & Min. Increment $\delta\hat{X}$ for a 10% Increase in Parameters α_1 & β_2

	Lower Order Observer Eigenvalues at (-10)	Lower Order Observer Eigenvalues at (-20)	Lower Order Observer Eigenvalues at (-40)	Full Order Eigenvalues at (-10,-15)
$\delta\hat{X}_2(\alpha_1)$	0.0	0.0	0.0	0.8×10^{-2}
	-1.0×10^{-2}	-0.95×10^{-2}	-0.9×10^{-2}	-2.8×10^{-2}
$\delta\hat{X}_2(\beta_2)$	1.1×10^{-2}	1.05×10^{-2}	1.0×10^{-2}	0.2×10^{-1}
	0.0	0.0	0.0	-0.84×10^{-1}

The nominal value of these parameters are

$$a_1^o = 2, \ a_2^o = 3, \ \beta_2^o = 2.$$

a_1 is considered to be a constant, therefore the system has two parameters, a_2 and β_2.

The system has to be stabilized by state feedback, in such a way that the eigenvalues of the system are set at (-3,-4). The state vector X_2 has to be estimated.

The procedure as adopted in Example 5 is followed in this example.

Results: The plots of the deviations of the estimated state, for a ten percent increase in parameter values is recorded in Figures 3.27 to 3.30. The maximum and minimum deviations for three sets of observer eigenvalues, (-17), (-23), and (-37) were tabulated in Table 3.3. Also the maximum and minimum deviations of $\hat{X}_2$ for the full order observer were tabulated in Table 3.3.

3.6.4. Effects on Sensitivity by the Eigenvalues of the Lower Order Observer

The eigenvalues of the lower order observer have an effect on the sensitivity of the lower order observer. This conclusion is drawn from the observations made in Tables 3.1, 3.2, and 3.3. In all the three examples, it has been noticed that there exists a pattern in the deviations due to increasing (negatively) eigenvalues. The eigenvalues considered were allowable, that is, they fulfill the necessary conditions as defined for the Luenberger observer.

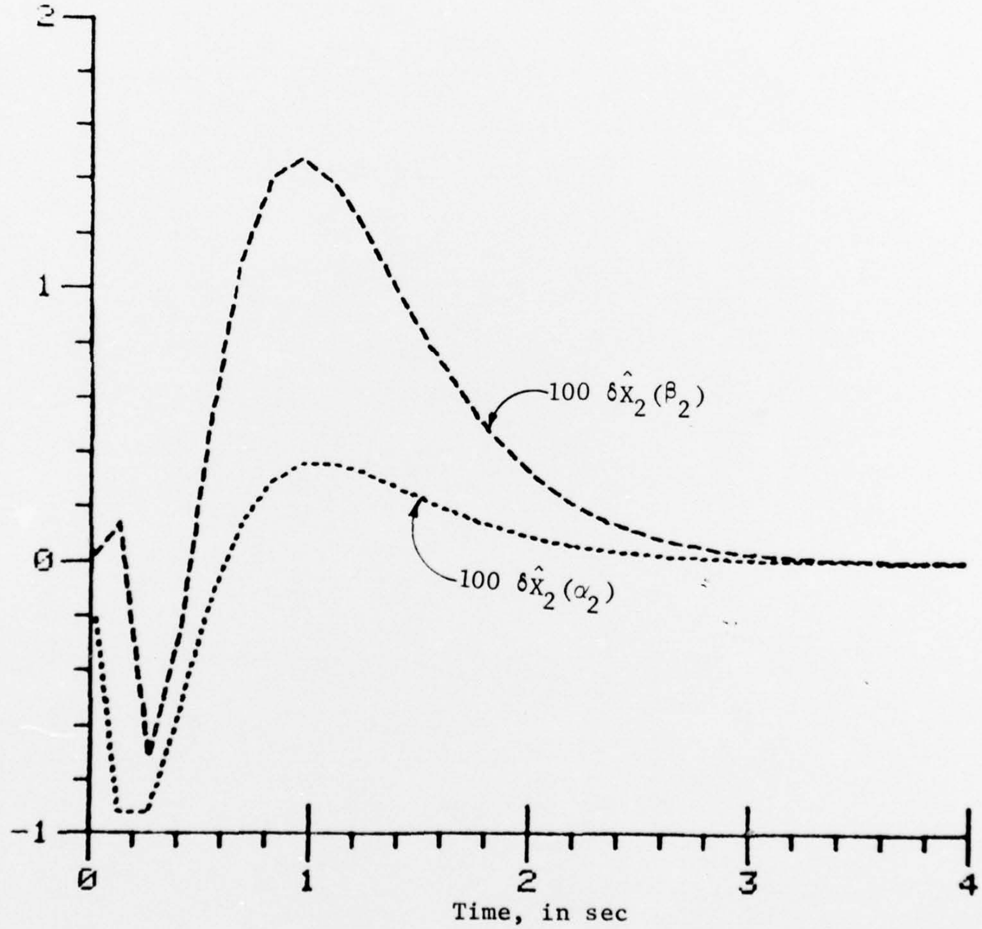

Figure 3.27 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 7. Observer eigenvalue at (-23).

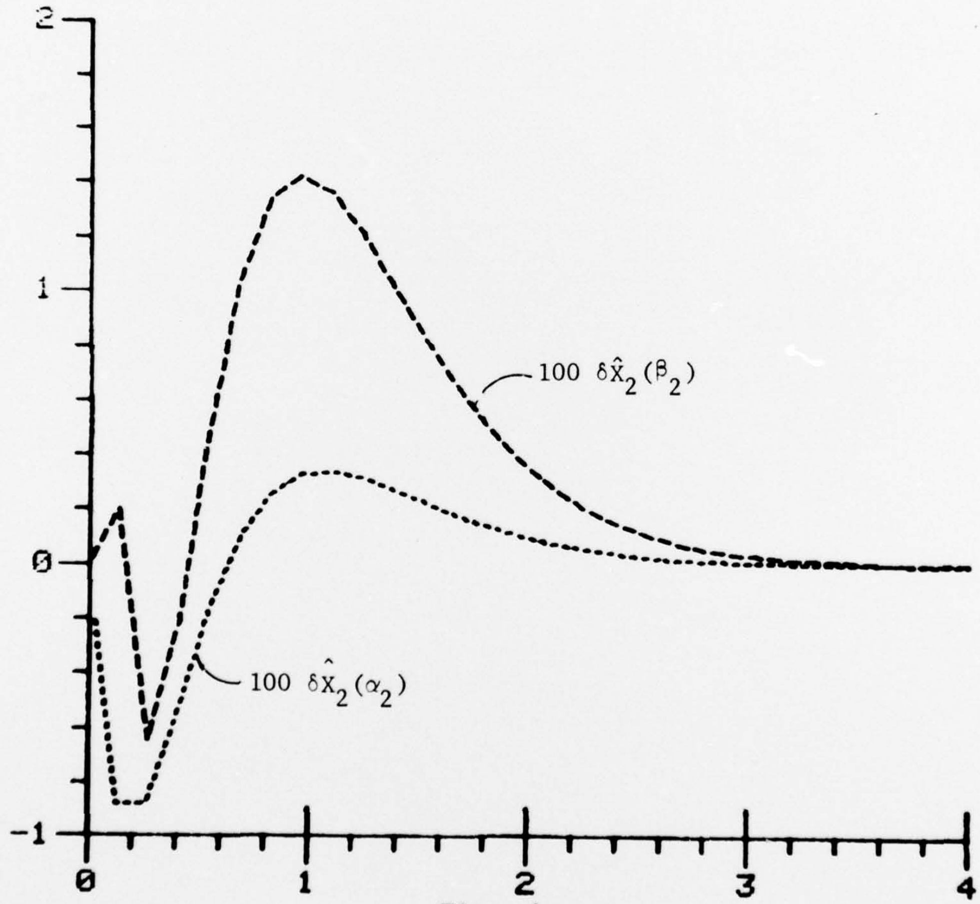

Figure 3.28 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 7. Observer eigenvalues at (-23).

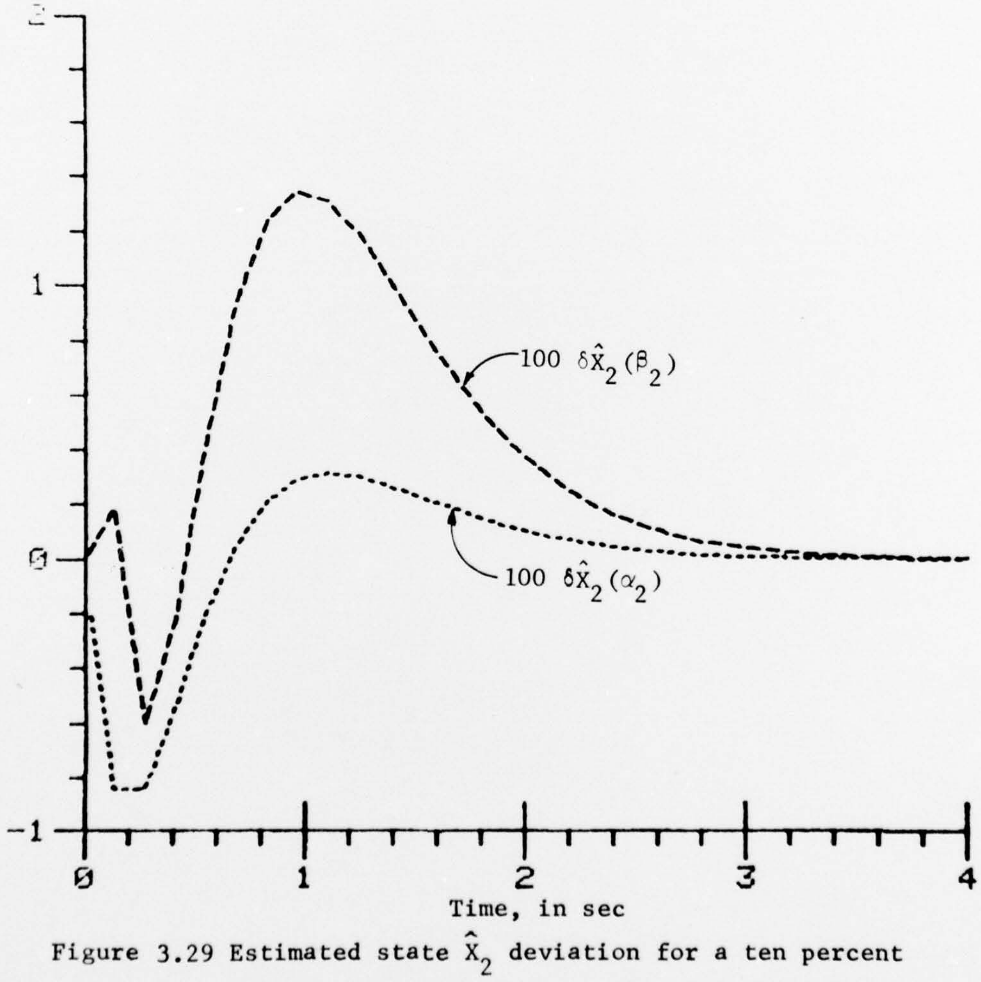

Figure 3.29 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for Example 7. Observer eigenvalue at (-37).

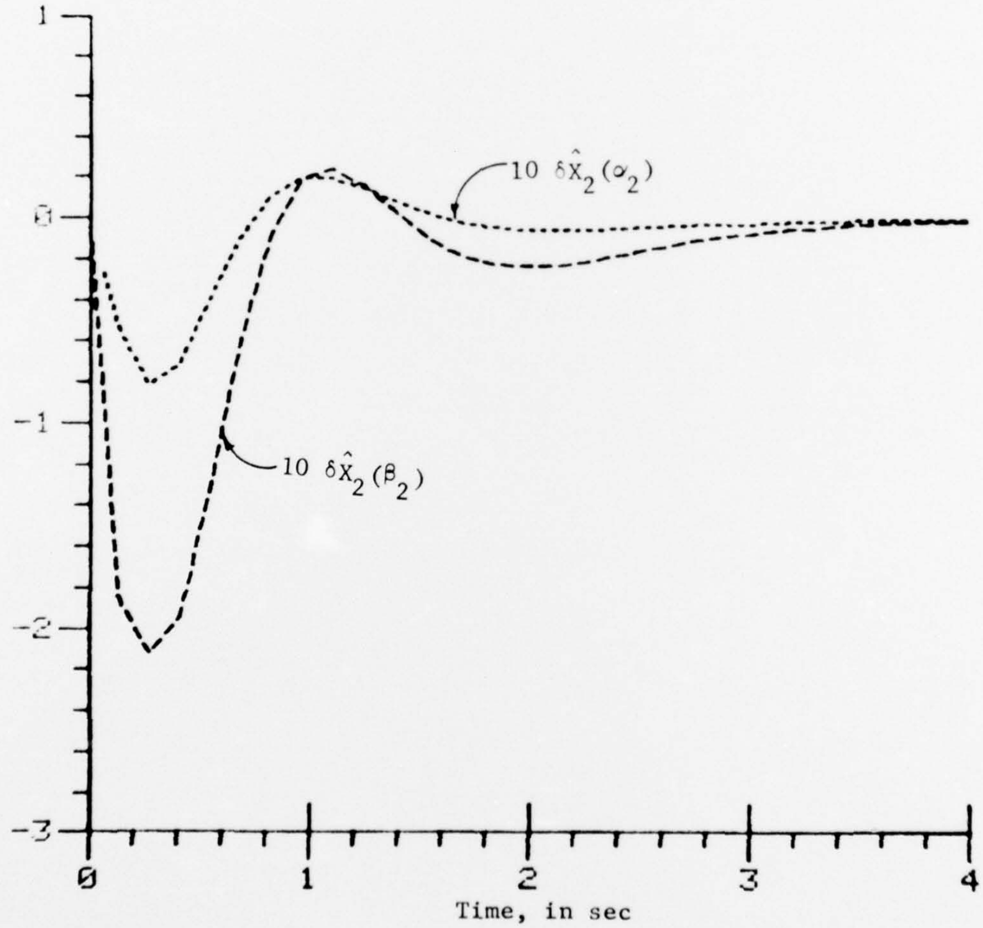

Figure 3.30 Estimated state $\hat{X}_2$ deviation for a ten percent increase in parameters, for the Full Order Observer of Example 7.

Table 3.3

For Example 7

Max. & Min. Increment $\delta\hat{X}$ for a 10% Increase in Parameters α_2 & β_2

		Lower Order Observer Eigenvalues at (-17)	Lower Order Observer Eigenvalues at (-23)	Lower Order Observer Eigenvalues at (-37)	Full Order Observer at (-15,-17)
$\delta\hat{X}_2(\alpha_2)$	max	0.32×10^{-2}	0.3×10^{-2}	0.28×10^{-2}	2.0×10^{-2}
	min	-0.95×10^{-2}	-0.9×10^{-2}	-0.85×10^{-2}	-6.0×10^{-2}
$\delta\hat{X}_2(\beta_2)$	max	1.5×10^{-2}	1.4×10^{-2}	1.3×10^{-2}	2.5×10^{-2}
	min	-0.7×10^{-2}	-0.65×10^{-2}	-0.6×10^{-2}	-1.35×10^{-1}

It has been noticed that, as the eigenvalues of the lower order observer become more negative than the previous eigenvalues, the deviations, maximum and minimum, in the estimated states are reduced. Thus it is concluded that if given a set of allowable eigenvalues, the most negative of these eigenvalues will make the lower order observer less dependent on the parameters of the system.

3.6.5. Comparison Between the Sensitivity of the Lower Order Observers and the Full Order Observers

In all the three examples, it has been noticed that the full order observer is highly sensitive as compared to the lower order observer. The Tables 3.1, 3.2, and 3.3 are a proof to this effect. The magnitude between the sensitivity of the two types of observer is also large. Thus, compelling us to conclude that the lower order observer is less sensitive to parameter changes than a full order observer. Note that the eigenvalues of both the observers were designed to be identical or very close to the other.

This concluded our experimental results. The DEC-10 computer was used to do the computation needed. The type of problems used are some typical types and were chosen to illustrate some aspects of sensitivity of the lower order observers.

4. CONCLUSIONS

This study is divided into two parts. The first was to develop the sensitivity model for the lower order observer, and that was carried out in Chapter 2. The second part was to analyze the sensitivity of the lower order observer. The experimentation and results were carried out in Chapter 3. Now we shall comment on the total study and on the subject of sensitivity of lower order observers, in general. It must be noted at this point that our conclusions are drawn, in the case of the sensitivity analysis from the results of seven examples that were analyzed theoretically and with the help of the DEC-10 computer.

The sensitivity model developed is a lower order, n-dimensional model. In certain cases the dimensions of the model is (n-r). All the sensitivity functions are obtained from this model. Thus we have succeeded in reducing the dimensions of the model.

The sensitivity of the lower order observers has given some insight into the design of these observers. The observed properties are the following. Not all systems can have a reasonable, parameter dependent lower order observer. There are other systems that can have lower order observers only under certain conditions. In general, systems may be able to have a reasonably good parameter dependent lower order observer that will estimate the states of the system without excessive errors in the estimation. Moreover, the eigenvalues of the lower order observer of a system, effects the sensitivity of the lower order observers. From a set of eigenvalues that satisfy the necessary conditions for designing the parameter independent lower order

observer, the most negative eigenvalues will make the lower order observer, least dependent on the parameters of the system. Therefore when designing a lower order observer it will be useful to keep this point in mind in order to design a good estimator.

The lower order observer is less dependent on the system parameters than the full order observers. Therefore, the lower order observer must be used wherever possible because it is less dependent on the parameters of the system and therefore will estimate better.

This concludes our study of the sensitivity of the lower order observer. All the conclusions reached are due to some theoretical and some experimental results. The Luenberger Observer was used to estimate the states.

It must be mentioned here that it is possible that the experimentation on other models of lower order observers may deviate from the conclusions reached in this thesis. There is no proof that these conclusions are true, but experimental results have shown that there may be some truth to our conclusions.

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